Finiteness properties of functor categories

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Notation

Let $\mathcal{C}$ be a small category and $\mathcal{E}$ a category. We denote by $\mathbf{Fct}(\mathcal{C}, \mathcal{E})$ the category of functors from $\mathcal{C}$ to $\mathcal{E}$, the arrows being natural transformations.

We are especially interested with the case where $\mathcal{E}$ is an abelian category (a module category, in general), sometimes also by the case where $\mathcal{E}$ is the category **Set** of sets.

Notation

Let $\mathcal{C}$ be a small category and k a ring. We denote by $\mathcal{F}(\mathcal{C}; k)$ the functor category $\mathbf{Fct}(\mathcal{C}, k - \mathbf{Mod})$, where $k - \mathbf{Mod}$ is the category of left k-modules.

Proposition

For any small category $\mathcal{C}$ and any abelian category $\mathcal{E}$, $\mathbf{Fct}(\mathcal{C}, \mathcal{E})$ *is an abelian category where exactness is tested pointwise. For any ring k, $\mathcal{F}(\mathcal{C}; k)$ is a nice abelian category (Grothendieck category).*

Notation

Let $\mathcal{C}$ be a small category, k a ring and x an object of $\mathcal{C}$. We denote by $P_x^{\mathcal{C}}$ the following object of $\mathcal{F}(\mathcal{C};k)$:

$$\mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(x,-)} \mathbf{Set} \xrightarrow{k[-]} k-\mathbf{Mod}$$

where $k[-]$ is the k-linearisation functor.

Lemma (Linearised Yoneda lemma)

$$\mathrm{Hom}_{\mathcal{F}(\mathcal{C};k)}(P_x^{\mathcal{C}}, F) \simeq F(x).$$

Corollary

The functors $P_x^{\mathcal{C}}$ are projective and finitely generated. When x runs over a skeleton of $\mathcal{C}$, the functors $P_x^{\mathcal{C}}$ generate the abelian category $\mathcal{F}(\mathcal{C};k)$.

We begin with combinatorial examples.

Notation

- We denote by Δ the category of non-empty finite ordinals.
- We denote by Θ the category whose objects are finite sets $\mathbf{n} := \{1, \ldots, n\}$ ($n \in \mathbb{N}$) and whose morphisms are *injective* functions.
- We denote by Ω the category with the same objects as Θ and *surjective* functions as morphisms.
- We denote by Γ the category with objects the finite *pointed* sets $\{0, \ldots, n\}$ (with 0 as base point) and with functions preserving the base point as morphisms.

We go on with linear examples.

Notation

Let R be a ring.

1. We denote by $\mathbf{P}(R)$ the category whose objects are the free left R-modules of finite rank R^n ($n \in \mathbb{N}$) and whose morphisms are R-linear maps.
2. We denote by $\mathbf{S}(R)$ the category with the same objects but with split R-monomorphisms, the splitting being given in the structure, as morphisms, that is:

$$\mathrm{Hom}_{\mathbf{S}(R)}(U,V) = \{ (f,g) \in \mathrm{Hom}_{\mathbf{P}(R)}(U,V) \times \mathrm{Hom}_{\mathbf{P}(R)}(V,U) | \, g \circ f = \mathrm{Id}_U \}.$$

The following functor category is of special interest.

Notation

Let k be a finite field. We denote by $\mathcal{F}(k)$ the category $\mathcal{F}(\mathbf{P}(k), k-\mathbf{Mod})$ of functors from finite dimensional vector spaces over k to vector spaces over k.

- **Algebraic topology (I)**: $\mathbf{Fct}(\Delta^{op}, \mathbf{Set})$ category of simplicial sets is a standard model for homotopy theory of spaces, $\mathcal{F}(\Delta^{op}; k)$ gives rise to the Dold-Kan correspondence...
- **Algebraic topology (II)**: if p is any prime number, since the 1980's deep relations between unstable modules over the Steenrod algebra mod. p and the category $\mathcal{F}(\mathbb{F}_p)$ have been proven and used. See [Henn-Lannes-Schwartz, The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects, *Amer. J. Math.* 1993].
- **Various areas of mathematics (especially topology and representation theory)**: a lot of mathematical objects carry naturally a structure of functor on the category Θ. See [Church-Ellenberg-Farb, FI-modules and stability for representations of symmetric groups, arXiv:1204.4533, to appear in *Duke*]. (FI-module=object of $\mathcal{F}(\Theta; k)$ for some ground ring k.)

- **For functors from Γ or** Ω: they can give models for stable homotopy theory, see [Segal, Categories and cohomology theories, *Topology* 1974] (here, the target category can be **Set** of simplicial sets). Pirashvili proved an equivalence of categories "à la Dold-Kan" $\mathcal{F}(\Gamma; k) \simeq \mathcal{F}(\Omega; k)$ and used these categories to recover in a conceptual way Hodge decomposition for Hochschild homology of commutative algebras and define and study higher Hochschild homology — see his paper [Hodge decomposition for higher order Hochschild homology, *Ann. Sci. École Norm. Sup.* 2000].

- **For functors from $\mathbf{S}(R)$ and $\mathbf{P}(R)$ (R being a ring): stable homology of linear groups**
  One has a functor "linear group"

$$GL : \mathbf{S}(R) \to \mathbf{Groups} \qquad R^n \mapsto GL_n(R)$$

$$(f,g) \in \mathrm{Hom}_{\mathbf{S}(R)}(U,V) \mapsto \big(\xi \in GL(U) \mapsto f\xi g + 1 - fg \in GL(V)\big).$$

For $F \in \mathcal{F}(\mathbf{S}(R); \mathbb{Z})$, we can form the sequence of graded abelian groups

$$\cdots \to H_*(GL_n(R); F(R^n)) \to H_*(GL_{n+1}(R); F(R^{n+1})) \to \dots$$

where the arrows are induced by the obvious maps $R^n \to R^{n+1}$ of $\mathbf{S}(R)$; call its colimit *stable homology* of general linear groups with coefficients into F and denote it by $H^{st}(GL(R); F)$.

Proposition

$H^{st}(GL(R); F)$ is naturally isomorphic to the functor homology $H_*(\mathbf{S}(R) \times GL_\infty(R); F)$, where the group $GL_\infty(R)$ acts trivially.

Problem: the computation of $H_*(\mathbf{S}(R); F)$ is almost always out of direct reach!

When F "comes from a polynomial bifunctor on $\mathbf{P}(R)$": one can see that this homology group is naturally isomorphic to some Hochschild homology group on the category $\mathbf{P}(R)$ (deep theorem of Scorichenko, 2000, unpublished). It makes computable several extremely hard groups of stable homology of general linear groups on finite fields or on the ring $\mathbb{Z}$ of integers.

1. Functor categories: examples, motivations, questions
   - Notations, first properties
   - Examples of interesting source categories
   - Motivations for looking at these functor categories
   - Main general questions on functor categories $\mathcal{F}(\mathcal{C}; k)$

2. The noetherian property for functors
   - Sam-Snowden approach (I): change the source category
   - Sam-Snowden approach (II): Gröbner categories

3. Polynomial functors
   - Classical setting
   - A more general setting

- **Related to representation theory**: describe simple, indecomposable projective or injective functors.

Example

Let k be a finite prime field. One has classical explicit bijections

$$\bigsqcup_{n\in\mathbb{N}}\{\text{iso. classes of simple } k[GL_n(k)]-\text{modules}\} \simeq$$

$$\{\text{iso. classes of simple objects of } \mathcal{F}(k)\} \simeq$$

$$\bigsqcup_{n\in\mathbb{N}}\{\text{iso. classes of simple } k[\mathfrak{S}_n]-\text{modules}\}.$$

But there are huge indecomposable injective functors in $\mathcal{F}(k)$, whose study can not be brought back to classical representation theory, and that are very mysterious.

- **Homological algebra**: compute Ext-groups, Tor-groups or related objects in $\mathcal{F}(\mathcal{C};k)$.
  This question has been very much studied in $\mathcal{F}(k)$, where k is a finite field [Franjou, Friedlander, Lannes, Schwartz, Scorichenko, Suslin...], also a little in $\mathcal{F}(\mathbf{P}(\mathbb{Z});\mathbb{Z})$ [Franjou-Pirashvili] or in $\mathcal{F}(\Gamma;\mathbb{Q})$ [Pirashvili].
  The setup of functor categories gives a lot of nice specific tools (play with different source categories, use adjunctions...) which are not available for group (co)homology, for example.
  (Group (co)homology is a particular case of functor (co)homology, because $\mathcal{F}(\underline{G};k)$, where $\underline{G}$ is the category with one object associated to a group G, is nothing but the category of representations of G with coefficients in k.)

- **Global structure and finiteness properties**:

Proposition

Let $\mathcal{C}$ be a small category and k a left-noetherian ring. The following conditions are equivalent.

1. *The abelian category $\mathcal{F}(\mathcal{C};k)$ is locally noetherian.*
2. *Each subfunctor of a finitely generated functor of $\mathcal{F}(\mathcal{C};k)$ is finitely generated.*
3. *Each finitely generated functor of $\mathcal{F}(\mathcal{C};k)$ has a resolution by finitely generated projective functors.*
4. *For all object x of $\mathcal{C}$, the functor $P_x^{\mathcal{C}}$ of $\mathcal{F}(\mathcal{C};k)$ is noetherian.*

Definition

Let $\mathcal{C}$ be a small category. We say that $\mathcal{C}$ has the property of finiteness for functors ((FF) for short) if the previous conditions hold for all left-noetherian ring k.

An other important question is to understand the quotients of the Krull filtration. Remember the definition, following Gabriel:

Definition

Let $\mathcal{A}$ a nice abelian (Grothendieck) category. We define an increasing sequence of localising (that is: thick and stable under colimits) subcategories $(\mathcal{K}_n(\mathcal{A}))_{n\in\mathbb{Z}}$ of $\mathcal{A}$ by:

- $\mathcal{K}_n(\mathcal{A})$ is reduced to null objects for $n < 0$;
- for $n \geq 0$, $\mathcal{K}_n(\mathcal{A})/\mathcal{K}_{n-1}(\mathcal{A})$ is the localising subcategory of $\mathcal{A}/\mathcal{K}_{n-1}(\mathcal{A})$ generated by simple objects.

An object of $\mathcal{A}$ is said of *Krull dimension* $\leq n$ if it belongs to $\mathcal{K}_n(\mathcal{A})$. It is said to be *noetherian of type n* if is noetherian and of Krull dimension n.

A basic example is the category of abelian groups, which has Krull dimension 1: $\mathcal{K}_0(\mathbf{Ab})$ consists of torsion abelian groups (denote this category by $\mathbf{Ab}_{tor}$), and $\mathbf{Ab} = \mathcal{K}_1(\mathbf{Ab})$, the quotient category $\mathbf{Ab}/\mathbf{Ab}_{tor}$ being equivalent to $\mathbb{Q}$-vector spaces through the rationalisation functor.

Another (too) simple example is the following: for any abelian category $\mathcal{A}$, if $\mathcal{A}$ is locally noetherian, then $\mathbf{Fct}(\Delta^{op}, \mathcal{A})$ is also locally noetherian; moreover this functor category has the same Krull dimension as $\mathcal{A}$. It follows from Dold-Kan correspondence.

Conjecture (Jean Lannes and Lionel Schwartz, late eighties)

Let k be a finite field. Then $\mathcal{F}(k)$ is a locally noetherian category.

Remark

It is easy to see that it is wrong for any infinite field.

Theorem (Sam-Snowden, 2014)

The previous conjecture holds. More generally, for any finite ring R, the category $\mathbf{P}(R)$ has the property (FF).

Theorem (Putman-Sam, 2014)

The previous conjecture holds. Moreover, for any finite commutative ring R, the category $\mathbf{S}(R)$ has the property (FF).

The conjecture is equivalent to the following statement: for any $n \in \mathbb{N}$, the functor

$$P_n := P_{k^n}^{\mathbf{P}(k)} : V \mapsto k[V^n]$$

is noetherian.

The first works on the conjecture led to make the following stronger conjecture.

Conjecture

For any finite field k and integer $n \geq 0$, the functor P_n is noetherian of type n.

- $n = 0$: $P_0 = k$ constant functor is simple, so noetherian of type 0.
- $n = 1$: the strong conjecture is not very hard to prove, in fact one can completely describe the lattice of subfunctors of P_1. Let us say this for the simplest case, $k = \mathbb{F}_2$. Then P_1 decomposes has $\mathbb{F}_2 \oplus \bar{P}$, where $\bar{P}$ associates to a finite $\mathbb{F}_2$-vector space the augmentation ideal of its $\mathbb{F}_2$-group algebra. The functor $\bar{P}$ is *uniserial* (that is: its lattice of subobjects is totally ordered); more precisely, its non-zero subfunctors are the various powers of the augmentation ideal (the subquotients being the exterior powers, which are simple functors). So, $\bar{P}$ is not of finite lenght, but each strict quotient is of finite lenght, proving that it is noetherian of type 1.

- For $n = 2$ and $k = \mathbb{F}_2$: Geoffrey Powell [*JPAA*, 1998] proved that P_2 is noetherian of type 2. It is a deep result requiring the introduction of several specific tools in the categories $\mathcal{F}(k)$. One can not describe completely the lattice of subfunctors of P_2.
- For $n = 3$ and $k = \mathbb{F}_2$: I proved [*Ann. Institut Fourier*, 2009] that P_3 is noetherian of type 3, by improving Powell's methods and introducing *grassmannian functors* [*Mém. SMF*, 2007], which permit to give a stronger conjecture describing the quotients of the Krull filtration of the category $\mathcal{F}(k)$.
- For $n \geq 4$, the strong form of the finiteness conjecture of $\mathcal{F}(k)$ remains fully open, even after Putman-Sam-Snowden work.

Theorem (Sam-Snowden)

The categories Ω^{op} and Γ^{op} have the property (FF).

This theorem, which implies the property (FF) for $\mathbf{P}(R)$ (for a finite ring R), was inspired by the following (quite easier) result.

Proposition (Church-Ellenberg-Farb-Nagpal, 2012)

The category Θ has the property (FF).

Definition

A category $\mathcal{C}$ is *directed* if

$$\forall x \in \mathrm{Ob}\,\mathcal{C} \qquad \mathrm{End}_{\mathcal{C}}(x) = \{\mathrm{Id}_x\}.$$

Notation

For any surjection $f \in \mathrm{Hom}_{\Omega}(\mathbf{i}, \mathbf{j})$, denote by $f^! : \mathbf{j} \to \mathbf{i}$ the injective map given by

$$f^!(r) := \min f^{-1}(\{r\}).$$

Observe that $(f \circ g)^! = g^! \circ f^!$ when $g^!$ is increasing, what allows to make the following definition.

Notation

The category Ω_{sh} is the (directed) subcategory of Ω of arrows f such that $f^!$ is increasing.

Proposition

Let $\iota : \Omega_{sh} \to \Omega$ be the inclusion functor. For any $n \in \mathbb{N}$, there is an isomorphism

$$\Omega(-, \mathbf{n}) \circ \iota \simeq \Omega_{sh}(-, \mathbf{n}) \times \mathfrak{S}_n$$

of functors $\Omega_{sh}^{op} \to \mathbf{Set}$.

Corollary

The property (FF) for Ω_{sh}^{op} implies the property (FF) for Ω^{op}.

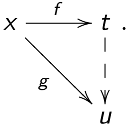
Hypothesis

Here $\mathcal{C}$ is a small skeletal directed category.

For any object x of $\mathcal{C}$, let

$$\mathcal{C}(x) := \bigsqcup_{t \in \mathrm{Ob}\,\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x, t)$$

endowed by the partial order relation $\leq_x$ defined by: $f \leq_x g$ if one can factorise g out of f:



Definition

A partial order relation $\leq$ on a set E is a *well partial order* if for any infinite sequence $(x_n)_{n\in\mathbb{N}}$ of E, there exist integers $i < j$ with $x_i \leq x_j$.

(So a well order is an order which is together a total order and a well partial order.)

Proposition

The functor $\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to$ **Set** *is noetherian if and only if* $\leq_x$ *is a well partial order on* $\mathcal{C}(x)$.

Definition (Sam-Snowden)

The category $\mathcal{C}$ is a *Gröbner* category if for any $x \in \mathrm{Ob}\,\mathcal{C}$:

1. $\leq_x$ is a well partial order on $\mathcal{C}(x)$;
2. there is an order $\preceq_x$ on $\mathcal{C}(x)$ such that:
   - $\preceq_x$ is a well order (in particuler, a total order);
   - for any maps $f, f' : x \to t$ and $g : t \to u$ of $\mathcal{C}$, the condition $f \prec_x f'$ implies $g \circ f \prec_x g \circ f'$.

Theorem (Sam-Snowden)

Any Gröbner category has the property (FF).

Theorem (Sam-Snowden)

The directed category Ω_{sh}^{op} *is a Gröbner category.*

Definition (Eilenberg-MacLane, early fifties)

Let $\mathcal{A}$ be an additive category, $\mathcal{E}$ an abelian category, $F : \mathcal{A} \to \mathcal{E}$ a functor and $d \in \mathbb{N}$.

- The *d-th cross-effect* of F is the functor $cr_d(F) : \mathcal{A}^d \to \mathcal{E}$ defined by

$$cr_d(F)(A_1, \dots, A_d) = Ker\left(F\Big(\bigoplus_{i=1}^{d} A_i\Big) \to \bigoplus_{i=1}^{d} F\Big(\bigoplus_{j \neq i} A_j\Big)\right).$$

- Let us say that F is *polynomial* of degree $\leq d$ if $cr_{d+1}(F) = 0$.

One has a natural decomposition

$$F\Big(\bigoplus_{i=1}^{d} A_i\Big) \simeq \bigoplus_{1 \leq i_1 < \cdots < i_k \leq d} cr_k(F)(A_{i_1}, \dots, A_{i_k}).$$

Let k be a field. We denote by $\mathcal{F}_d(k)$ the full subcategory of $\mathcal{F}(k)$ of polynomial functors of degree $\leq d$. It is a bilocalizing subcategory of $\mathcal{F}(k)$ — that is, it is thick and stable under limits and colimits.
If $F \in \mathrm{Ob}\,\mathcal{F}(k)$ takes finite-dimensional values, then F belongs to $\mathcal{F}_d(k)$ if and only if the function

$$\mathbb{N} \to \mathbb{N} \qquad n \mapsto \dim F(k^n)$$

is polynomial. The d-th tensor (or exterior, symmetric) power is a typical example of functor in $\mathcal{F}_d(k)$.

Proposition

Let k be a field, d a non-negative integer and $F \in \mathrm{Ob}\,\mathcal{F}_d(k)$. The following conditions are equivalent.

- *F is of finite lenght ;*
- *F is noetherian ;*
- *F is finitely generated ;*
- *F takes finite-dimensional values.*

If moreover k is finite, then these conditions are equivalent to the existence of a resolution of F by finitely-generated projective functors (in $\mathcal{F}(k)$), and all functors of finite lenght of $\mathcal{F}(k)$ are polynomial.

Reference: A. Djament and C. Vespa, *De la structure des foncteurs polynomiaux sur les espaces hermitiens*, arXiv 1308.4106.

Let $(\mathcal{C}, +, 0)$ be a small symmetric monoidal category whose unit 0 is an initial object and k a ring. For $x \in \mathrm{Ob}\,\mathcal{C}$, let τ_x be the endofunctor of precomposition by $x + -$, in $\mathcal{F}(\mathcal{C}; k)$. Let δ_x be the cokernel of the natural transformation $\mathrm{Id} = \tau_0 \to \tau_x$ (δ_x is the *difference functor* associated to x).

Definition

One says that $F \in \mathrm{Ob}\,\mathcal{F}(\mathcal{C}; k)$ is *strongly polynomial* of (strong) degree $\leq d$ if for any objects $x_0, x_1, \dots, x_d$ in $\mathcal{C}$, one has

$$\delta_{x_0}\delta_{x_1}\dots\delta_{x_d}(F) = 0.$$

(This definition is equivalent to the classical one if $\mathcal{C}$ is an additive category endowed with the categorical sum.)

Problem: this notion is not stable under subfunctor.
A solution: change the definition by working in the following quotient category instead of $\mathcal{F}(\mathcal{C};k)$.
Denote by SN the localising subcategory of $\mathcal{F}(\mathcal{C};k)$ of functors F which are the union over the objects x of $\mathcal{C}$ of the kernel of the canonical maps $F \to \tau_x(F)$ and by $\mathbf{St}(\mathcal{C};k)$ the quotient category. The endofunctors δ_x of $\mathcal{F}(\mathcal{C};k)$ induce *exact* endofunctors of $\mathbf{St}(\mathcal{C};k)$.

Definition

One says that $X \in \mathrm{Ob}\,\mathbf{St}(\mathcal{C};k)$ is *polynomial* of degree $\leq d$ if for any objects $x_0, x_1, \dots, x_d$ in $\mathcal{C}$, one has

$$\delta_{x_0}\delta_{x_1}\dots\delta_{x_d}(X) = 0.$$

One says that an object of $\mathcal{F}(\mathcal{C};k)$ is *weakly polynomial* of (weak) degree $\leq d$ if its image in $\mathbf{St}(\mathcal{C};k)$ is polynomial of degree $\leq d$.

Here we get a well-behaved notion: the full subcategory of weakly polynomial functors of degree $\leq d$ is bilocalising.

From now, our source category will be $\mathbf{S}(\mathbb{Z})$, with the monoidal structure induced by the direct sum of abelian groups.

In my preprint *Des propriétés de finitude des foncteurs polynomiaux* (arXiv 1308.4698), the following is proven.

Theorem

Let $F : \mathbf{S}(\mathbb{Z}) \to \mathbf{Ab}$ be a finitely generated weakly polynomial functor. There exists an integer $n \geq 0$ such that the restriction of F to the full subcategory $\mathbf{S}(\mathbb{Z})_{\geq n}$ of $\mathbf{S}(\mathbb{Z})$ of abelian groups whose rank is $\geq n$ is noetherian.

(F can be *not* noetherian, because the group ring $\mathbb{Z}[GL_n(\mathbb{Z})]$ is not noetherian for $n \geq 2$.)

One reason of interest for the category $\mathcal{F}(\mathbf{S}(\mathbb{Z});\mathbb{Z})$ is the following. Let I be a ring *without unit* and, for $n \in \mathbb{N}$, $GL_n(I)$ be the associated "linear" group (which is in fact a congruence group), that is:

$$GL_n(I) := Ker\,(GL_n(\tilde{I}) \twoheadrightarrow GL_n(\mathbb{Z})),$$

$\tilde{I}$ ($= \mathbb{Z} \oplus I$ as an abelian group) being the ring obtained by formally adding a unit to I. One shows easily that $\tilde{I}^t \mapsto GL_t(I)$ defines a functor $\mathbf{S}(\tilde{I}) \to \mathbf{Grp}$ and that it induces a functor $H_n(GL(I)) : \mathbf{S}(\mathbb{Z}) \to \mathbf{Ab}$ for each integer n.

Conjecture

For any ring without unit I and any integer $n \in \mathbb{N}$, the functor $H_n(GL(I))$ of $\mathcal{F}(\mathbf{S}(\mathbb{Z});\mathbb{Z})$ is weakly polynomial of degree $\leq 2n$.

Suslin proved a particular case of this statement in his work on excision in integer algebraic K-theory [*Trudy Mat. Inst. Steklov*, 1995].