Noetherianity and homological finiteness of polynomial functors (III)

Aurélien DJAMENT

CNRS, LAGA, Villetaneuse, France

November 2023

Last talk of a mini-course to Vietnam Institute for Advanced Study in Mathematics (VIASM), in the framework of the program *Algebraic Topology Activity 2023*.

After recent joint works with Antoine Touzé.

Plan of the talk

1. Property fp_∞ for ordinary polynomial functors
   - Case of additive functors
   - Case of polynomial functors (under a noetherian assumption) — statement
   - Proof
2. Property fp_∞ for strict polynomial functors
3. The main theorem
4. Proof of the key result on tensor products

A theorem of preservation of property fp_∞

The following theorem is old but it has been not published for a long time; see Scholze, *Consended mathematics* (appendix to lecture IV).

Theorem (Breen, Deligne...)

Let $\mathcal{A}$ be a small additive category. The inclusion functor $\mathbf{Add}(\mathcal{A};\mathbb{Z}) \to \mathcal{F}(\mathcal{A};\mathbb{Z})$ preserves property fp_n for each $n \in \mathbb{N} \cup \{\infty\}$.

This theorem really deals with abelian groups (after, precompose by a functor $\mathcal{A}(a,-)$); it is based on classical but non-trivial results of algebraic topology on the homology of Eilenberg-MacLane spaces.

An extension to polynomial functors

Theorem (D.-Touzé)

Let $\mathcal{A}$ be a small additive category and $d \in \mathbb{N}$. Let us also assume that the category $\mathcal{P}ol_d(\mathcal{A};\mathbb{Z})$ is locally noetherian. Then each finitely-generated functor of $\mathcal{P}ol_d(\mathcal{A};\mathbb{Z})$ satisfies property fp_∞ in $\mathcal{F}(\mathcal{A};\mathbb{Z})$.

(From $d = 2$, it is not hard to see that the conclusion may fail if one removes the assumption of local noetherianity.)

This result is proved by induction on d. We will sketch this proof.

Proof (I): an essential result on tensor products

The hardest part of the proof of the previous theorem from the additive case is to prove the following result, that we will prove at the end of this talk.

Theorem (D.-Touzé)

Let $\mathcal{A}$ be a small additive category, $n \in \mathbb{N} \cup \{\infty\}$ and $\mathfrak{P}$ a class of objects of $\mathcal{F}(\mathcal{A}; \mathbb{Z})$ satisfying the following conditions:

1. *$\mathfrak{P}$ is stable under subobject;*
2. *each object of $\mathfrak{P}$ is noetherian;*
3. *$\mathfrak{P} \subset fp_n(\mathcal{F}(\mathcal{A}; \mathbb{Z}))$.*

Then each tensor product of elements of $\mathfrak{P}$ satisfies fp_n.

Remark

It is clear that the tensor product of functors of $\mathcal{F}(\mathcal{A};\mathbb{Z})$ which satisfy fp_n and *have values without torsion over* $\mathbb{Z}$ satisfies fp_n (look at the tensor product of finitely-generated projective functors!); the whole difficulty comes from torsion.
Without additional assumption, a tensor product of functors (even additive), of $\mathcal{F}(\mathcal{A};\mathbb{Z})$ satisfying fp_2 may not satisfy fp_2.

By combining the previous theorem with the one of the beginning (Breen-Deligne...), one obtains:

Corollary

If the category $\mathbf{Add}(\mathcal{A},\mathbb{Z})$ *is locally noetherian, then each tensor product of finitely-generated additive functors satsifies property* fp_∞ *in* $\mathcal{F}(\mathcal{A};\mathbb{Z})$.

Proof (II): diagonalisation of cross-effects satisfies fp_∞

We assume that the assumptions of the theorem are fulfilled: $\mathcal{A}$ is additive such that $\mathcal{P}ol_d(\mathcal{A},\mathbb{Z})$ is locally noetherian, and F denotes a finitely-generated functor of $\mathcal{P}ol_d(\mathcal{A},\mathbb{Z})$.

Lemma

The functors $cr_d : \mathcal{F}(\mathcal{A};\mathbb{Z}) \to \mathcal{F}(\mathcal{A}^d;\mathbb{Z})$ and $\Delta_d : \mathcal{F}(\mathcal{A}^d;\mathbb{Z}) \to \mathcal{F}(\mathcal{A};\mathbb{Z})$ (precomposition by iterated diagonal) preserve property fp_n for each n.

This follows from the adjunction (on both sides) of these functors and that they are exact (so the adjunction extensions to extension groups) and commute to colimits.

In particular, $cr_d(F)$ is a finitely-generated multifunctor of $\mathbf{Add}_d(\mathcal{A};\mathbb{Z})$ (category of d-multifunctors which are additive with respect to each entry).

The category $\mathbf{Add}_d(\mathcal{A};\mathbb{Z})$ is locally noetherian because $\mathcal{P}ol_d(\mathcal{A};\mathbb{Z})$ is supposed to be locally noetherian, so $\Sigma\mathbf{Add}_d(\mathcal{A};\mathbb{Z}) \simeq \mathcal{P}ol_d(\mathcal{A};\mathbb{Z})/\mathcal{P}ol_{d-1}(\mathcal{A};\mathbb{Z})$ is too, and one gets the results by finiteness of the symmetric group $\mathfrak{S}_d$.

The category $\mathbf{Add}_d(\mathcal{A};\mathbb{Z})$ is moreover generated by finitely-generated projective functors $\mathcal{A}(a_1,-)\boxtimes\cdots\boxtimes\mathcal{A}(a_d,-)$, where the a_i's are objects of $\mathcal{A}$.
So, $cr_d(F)$ has a projective resolution whose each term is a *finite* direct sum of functors of this kind, so $\Delta_d(cr_d(F))$ has a resolution whose terms are finite direct sums of functors of the shape $\mathcal{A}(a_1,-)\otimes\cdots\otimes\mathcal{A}(a_d,-)$.

By our corollary, it follows that $\Delta_d(cr_d(F)) \in fp_\infty(\mathcal{F}(\mathcal{A};\mathbb{Z}))$.

Proof (III): passing to coinvariants under the action of $\mathfrak{S}_d$

The bar resolution for the group $\mathfrak{S}_d$ gives a complex of the shape

$$\cdots \to \mathbb{Z}[\mathfrak{S}_d^n] \otimes \Delta_d(cr_d(F)) \to \cdots \to \mathbb{Z}[\mathfrak{S}_d] \otimes \Delta_d(cr_d(F)) \to \Delta_d(cr_d(F))$$

in $\mathcal{F}(\mathcal{A};\mathbb{Z})$ which has the following properties:

1. its homology in degree 0 is $\Delta_d(cr_d(F))_{\mathfrak{S}_d}$;
2. its homology in each positive degree belongs to $\mathcal{P}ol_{d-1}(\mathcal{A};\mathbb{Z})$ (consequence of Pirashvili's structure theorem) and is finitely-generated (as a subquotient of a noetherian functor);
3. so, the induction assumption implies that this homology belongs to $fp_\infty(\mathcal{F}(\mathcal{A};\mathbb{Z}))$ in each positive degree;
4. each term ot the complex belongs to $fp_\infty(\mathcal{F}(\mathcal{A};\mathbb{Z}))$, as $\Delta_d(cr_d(F))$.

It implies formally that $\Delta_d(cr_d(F))_{\mathfrak{S}_d} \in fp_\infty(\mathcal{F}(\mathcal{A};\mathbb{Z}))$.

Proof (IV): end of the inductive step

By Pirashvili's theorem, on has an exact sequence

$$0 \to G \to \Delta_d(cr_d(F))_{\mathfrak{S}_d} \to F \to H \to 0$$

where G and H belong to $\mathcal{P}ol_{d-1}(\mathcal{A};\mathbb{Z})$. Moreover, H is finitely-generated (as a quotient of F), and G too (as a subquotient of $\Delta_d(cr_d(F))$, which is noetherian). So, this induction hypothesis implies that G and H belong to $fp_\infty(\mathcal{F}(\mathcal{A};\mathbb{Z}))$. As $\Delta_d(cr_d(F))_{\mathfrak{S}_d}$ too, it follows that $F \in fp_\infty(\mathcal{F}(\mathcal{A};\mathbb{Z}))$, whence the theorem.

Property fp_∞ for strict polynomial functors

Remember that $i_d : \mathcal{P}_{d;\mathbb{Z}}(\mathcal{A};k) \to \mathcal{F}(\mathcal{A};k)$ denotes the forgetful functor.

Theorem (D.-Touzé)

Let $\mathcal{A}$ be a small additive category, k a commutative noetherian ring and $d \in \mathbb{N}$. Let us assume that the categories $\mathcal{P}_{d;\mathbb{Z}}(\mathcal{A};k)$ and $\mathbf{Add}(\mathcal{A};\mathbb{Z})$ are locally noetherian.
Then for each finitely-generated functor F of $\mathcal{P}_d(\mathcal{A};k)$, $i_d(F)$ belongs to $fp_\infty(\mathcal{F}(\mathcal{A};k))$.

Corollary

Let k and R be commutative noetherian rings. Let us assume that the k-algebra $R_k := R \otimes_{\mathbb{Z}} k$ is essentially finitely-generated. Then for each $d \in \mathbb{N}$ and each finitely-generated functor F of $\mathcal{P}_{d;\mathbb{Z}}(R,k)$, the functor $i_d(F)$ belongs to $fp_\infty(\mathcal{F}(R,k))$.

The corollary is a consequence of the theorem, of the noetherianity result of the previous talk and of the equivalence of categories $\mathcal{P}_{d;\mathbb{Z}}(R,k) \simeq \mathcal{P}_{d;k}(R_k,k)$.

Proof of the theorem

It is enough to prove that for each $a \in \mathrm{Ob}\,\mathcal{A}$, the functor $\Gamma^d_{\mathbb{Z}}(\mathcal{A}(a,-)) \otimes_{\mathbb{Z}} k$ belongs to $fp_\infty(\mathcal{F}(\mathcal{A};k))$.

As $\mathcal{A}(a,-)$ is finitely-generated in the locally noetherian category $\mathbf{Add}(\mathcal{A};\mathbb{Z})$, its torsion subfunctor is finitely-generated, so there exists an integer $N > 0$ such that the torsion of $\mathcal{A}(a,-)$ is annihilated by N. Let us denote by $\mathbf{ab}_N$ the full subcategory of $\mathbf{Ab}$ on finitely-generated abelian groups whose torsion subgroup is annihilated by N. This category is equivalent to $\mathbf{P}(\mathrm{A}_N)$, where A_N denotes the endomorphism ring of the abelian group $\mathbb{Z} \oplus \bigoplus \mathbb{Z}/n$, where the direct sum is taken on positive divisors n of N. As the underlying additive group of A_N is finitely-generated, the category $\mathcal{P}ol_d(\mathbf{ab}_N;\mathbb{Z})$ is locally noetherian (cf. talk 2), so that the theorem of the beginning of this talk proves that the restriction to $\mathbf{ab}_N$ of $\Gamma^d_{\mathbb{Z}}$ belongs to $fp_\infty(\mathcal{F}(\mathbf{ab}_N;\mathbb{Z}))$.

As the functors $\Gamma^d_{\mathbb{Z}}$ and $\mathbb{Z}[-] : \mathbf{Ab} \to \mathbf{Ab}$ commute to *filtered* colimits, each finitely-generated projective resolution of the restriction of $\Gamma^d_{\mathbb{Z}}$ to $\mathbf{ab}_N$ extends in a resolution of its restriction to the full subcategory $\mathbf{Ab}_N$ of $\mathbf{Ab}$ abelian groups whose torsion is annihilated by N. By precomposing by $\mathcal{A}(a, -)$, one gets the wished result for $k = \mathbb{Z}$.

When k has no $\mathbb{Z}$-torsion, the conclusion follows by flat base-change. In the general case, one uses:

Lemma

Let k be a noetherian ring and F a noetherian functor of $\mathbf{Add}(\mathcal{A};\mathbb{Z})$. Then $\mathrm{Tor}_1^{\mathbb{Z}}(F, k)$ is a finitely-generated functor of $\mathbf{Add}(\mathcal{A}; k)$.

The conclusion is then gotten by formal arguments similar to the one that we are going to present later to prove the theorem on tensor products of functors satisfying fp_n.

Proof of the lemma

As the ring k is noetherian, the ideal of its $\mathbb{Z}$-torsion is finitely-generated, so it is annihilated by some integer $r > 0$. It follows that the natural map

$$\bigoplus_n \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, k) \otimes_{\mathbb{Z}} \mathrm{Tor}_1^{\mathbb{Z}}(F, \mathbb{Z}/n) \to \mathrm{Tor}_1^{\mathbb{Z}}(F, k)$$

of $\mathbf{Add}(\mathcal{A}; k)$ deduced from the functoriality of $\mathrm{Tor}_1^{\mathbb{Z}}$, where the direct sum is taken over the finite set of positive divisors of r, is surjective. The conclusion comes then because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, k) \hookrightarrow k$ is a finitely-generated k-module and $\mathrm{Tor}_1^{\mathbb{Z}}(F, \mathbb{Z}/n) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, F) \hookrightarrow F$ is a finitely-generated functor of $\mathbf{Add}(\mathcal{A}; \mathbb{Z})$, as k and F are noetherian.

Statement of the main theorem

Theorem (D.-Touzé)

Let R be an essentially finitely-generated commutative ring, k a commutative noetherian ring and F a finitely-generated polynomial functor of $\mathcal{F}(R,k)$. Then F is noetherian and satisfies property fp_∞.

Before explaining the proof of this theorem, which is short form all the previous resuts, we are going to give a few generalisations and examples.

A few improvements of the statement

Instead of assuming R to be commutative, one can only assume that the k-algebra $R_k := R \otimes_{\mathbb{Z}} k$ is essentially finitely-generated and that R is noetherian. One can also assume only that R is non-commutative, right noetherian, and that the k-algebra R_k is finite (i.e. finitely generated as a module) over an essentially finitely-generated k-algebra. The proof is the same.

If we assume R to be commutative, non necessarily noetherian, and that the k-algebra R_k is essentially finitely generated, the noetherian conclusion still holds, but not property fp_∞. The proof goes by reducing to the previous theorem by change of basis ring at the source, thanks to elementary commutative algebra.

Example: the case of a field

Corollary

Let k be a field and $d \geq 1$ an integer. The following statements are equivalent:

1. *k is a finitely-generated field;*
2. *the category $\mathcal{P}ol_d(k,k)$ is locally noetherian;*
3. *each finitely-generated functor of $\mathcal{P}ol_d(k,k)$ belongs to $fp_\infty(\mathcal{F}(k,k))$;*
4. *each finitely-generated functor of $\mathcal{P}ol_d(k,k)$ belongs to $fp(\mathcal{F}(k,k))$.*

In fact, the ring $k \otimes_{\mathbb{Z}} k$ is noetherian if and only if k is a finitely-generated field.

Proof of the main theorem

It goes by induction on the degree d of our finitely-generated polynomial functor F of $\mathcal{F}(R,k)$. One can assume $d>0$ and the statement proven for functors of degree $<d$.

The multifunctor $cr_d(F)$ is finitely-generated, as the functor $\Delta_d(cr_d(F))$. As $\Delta_d \circ cr_d \simeq i_d \circ \tilde{\Delta}_d \circ cr_d$, it follows that the *strict* polynomial functor $\tilde{\Delta}_d(cr_d(F))$ is finitely-generated in the locally noetherian (cf. talk 2) category $\mathcal{P}_{d;\mathbb{Z}}(R,k) \simeq \mathcal{P}_{d;k}(R_k,k)$. So its subfunctor $\tilde{\Delta}_d^{\mathfrak{S}_d}(cr_d(F))$ is finitely-generated in $\mathcal{P}_{d;\mathbb{Z}}(R,k)$, and $\Delta_d^{\mathfrak{S}_d}(cr_d(F)) \simeq i_d\big(\tilde{\Delta}_d^{\mathfrak{S}_d}(cr_d(F))\big)$ is finitely-generated in $\mathcal{F}(R,k)$.

Let us denote by T and X the kernel and cokernel of the unit $F \to \Delta_d^{\mathfrak{S}_d}(cr_d(F))$: T and X belong to $\mathcal{P}ol_{d-1}(R,k)$ by Pirashvili's theorem. Moreover, X is finitely-generated because it is a quotient of $\Delta_d^{\mathfrak{S}_d}(cr_d(F))$ which is finitely-generated. The induction hypothesis implies then that $X \in fp_\infty(\mathcal{F}(R,k)) \subset fp_2(\mathcal{F}(R,k))$. Moreover, $\Delta_d^{\mathfrak{S}_d}(cr_d(F)) \in fp_\infty(\mathcal{F}(R,k)) \subset fp_1(\mathcal{F}(R,k))$ by the theorem that we proved on strict polynomial functors (one knows that $\mathcal{P}_{d;\mathbb{Z}}(R,k)$ is locally noetherian, and $\mathbf{Add}(R,\mathbb{Z}) \simeq R$-$\mathbf{Mod}$ is locally noetherian because R is noetherian).
The exact sequence

$$0 \to T \to F \to \Delta_d^{\mathfrak{S}_d}(cr_d(F)) \to X \to 0$$

with F (resp. $\Delta_d^{\mathfrak{S}_d}(cr_d(F))$, X) fp_0 (resp. fp_1, fp_2) proves that T is finitely-generated.

The induction hypothesis implies then that T satisfies fp_∞. As $\Delta_d^{\mathfrak{S}_d}(cr_d(F))$ and X satisfy it too, the same exact sequence proves now that F satisfies fp_∞.

So each finitely-generated functor of $\mathcal{P}ol_d(R,k)$ belongs to $fp_\infty(\mathcal{F}(R,k))$ and is in particular finitely presented in $\mathcal{F}(R,k)$. It follows that such a functor F is noetherian: if G is a subfunctor of F, then F/G is finitely-generated, so finitely presented, in $\mathcal{F}(R,k)$, what implies that G is finitely-generated. It proves the theorem.

Proof of the key result on tensor product of fp_n-functors (I)

If F and G are functors of $fp_n(\mathcal{F}(\mathcal{A};\mathbb{Z}))$ (with $\mathcal{A}$ additive), one denotes by $\mathrm{Tor}(F,G)$ the functor $\mathcal{A} \xrightarrow{(F,G)} \mathbf{Ab} \times \mathbf{Ab} \xrightarrow{\mathrm{Tor}_1^{\mathbb{Z}}} \mathbf{Ab}$ of $\mathcal{F}(\mathcal{A};\mathbb{Z})$.

Lemma

Let F and G be functors of $\mathcal{F}(\mathcal{A};\mathbb{Z})$ and $n \in \mathbb{N} \cup \{\infty\}$. Let us assume that F and G have property fp_n. Then the functor $F \otimes G$ satisfies fp_n if and only if $\mathrm{Tor}(F,G)$ satisfies fp_{n-2}.

(By convention, property fp_i is empty for $i < 0$; in particular, a tensor product of finitely presented functors is finitely presented.)

The proof is formal from the fact that finitely-generated projectives in $\mathcal{F}(\mathcal{A};\mathbb{Z})$ generate this category, have $\mathbb{Z}$-flat values and are stable under tensor product.

Proof of the key result on tensor product of fp_n-functors (II): preliminaries on torsion

Let us remind that, for an integer $N > 0$, $\mathbf{Ab}_N$ denotes the full subcategory of **Ab** on abelian groups whose torsion subgroup is annihilated by N, and $\mathbf{ab}_N$ is its full subcategory on finitely-generated objects.

If V is an abelian group, one denotes by V_{tor} its torsion subgroup and, for $n \in \mathbb{N}$, by $V_{(n)}$ the subgroup of elements annihilated by n. If F is a functor of $\mathcal{F}(\mathcal{A};\mathbb{Z})$, one uses the notations F_{tor} and $F_{(n)}$ for postcomposition by the previous endofunctors of abelian groups.

One says that F has *bounded torsion* if there exists an integer $N > 0$ such that $F_{\mathrm{tor}} = F_{(N)}$ (ons says then that F has torsion bounded by N), that is, if F takes values in $\mathbf{Ab}_N$.
A noetherian functor has always bounded torsion.

Lemma

Let $N > 0$ be an integer and $T : \mathbf{Ab}_N\text{-}\mathbf{Mod} \times \mathbf{Ab}_N\text{-}\mathbf{Mod} \to \mathbf{Ab}$ a bifunctor which is additive with respect to each entry and commutes with filtered colimits. Let us assume also that $T(U, V)$ is finitely generated if U and V are finitely generated.
Then there exists a resolution of T by finite direct sums of functors of the shape $(U, V) \mapsto U_{(i)} \otimes V_{(j)}$, with $(i, j) \in \mathbb{N}^2$.

As everything commutes with filtered colimits, one can only deal with the restriction of T to $\mathbf{ab}_N \times \mathbf{ab}_N$. The category $\mathbf{Add}_2(\mathbf{ab}_N; \mathbb{Z})$ is locally noetherian, and its finitely generated objects are exactly bifunctors with finitely generated values.
So, T belongs to $fp_\infty(\mathbf{Add}_2(\mathbf{ab}_N; \mathbb{Z}))$. As $\mathbf{Add}_2(\mathbf{ab}_N; \mathbb{Z})$ is generated by finitely-generated projective bifunctors $(U, V) \mapsto U_{(i)} \otimes V_{(j)}$, for $(i, j) \in (\mathbb{N})^2$, the lemma is proved.

Proof of the key result on tensor product of fp_n-functors (III): the key property

Proposition

Let F, G be functors of $\mathcal{F}(\mathcal{A};\mathbb{Z})$ and $n, m \in \mathbb{N} \cup \{\infty\}$. Let us assume that:

1. *F and G satisfy fp_n;*
2. *F and G have bounded torsion;*
3. *for each integer $i > 0$, the functors $F_{(i)}$ and $G_{(i)}$ satisfy fp_m.*

Then:

(A) *the functor $F \otimes G$ satisfies $fp_{\min(n,m+2)}$;*

(B) *the functor $\mathrm{Tor}(F, G)$ satisfies $fp_{\min(n,m)}$;*

(C) *if $T : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$ is a bifunctor which, with respect to each entry, is additive and commutes with filtered colimits, and maps every pair of finitely-generated abelian on a finitely-generated abelian group, then $T \circ (F, G) \in fp_{\min(n,m)}(\mathcal{F}(\mathcal{A};\mathbb{Z}))$.*

Structure of the recursive proof of the key property

Let us consider, for $n \in \mathbb{Z} \cup \{\infty\}$, the following conditions:

$(A)_n$ If F and G are functors of $fp_n(\mathcal{F}(\mathcal{A};\mathbb{Z}))$, of bounded torsion, and that for each integer $i > 0$, $F_{(i)}$ and $G_{(i)}$ satisfy fp_{n-2}, then $F \otimes G$ satisfies fp_n.

$(B)_n$ If F and G are functors of $fp_n(\mathcal{F}(\mathcal{A};\mathbb{Z}))$, of bounded torsion, and that for each integer $i > 0$, $F_{(i)}$ and $G_{(i)}$ satisfy fp_n, then $\mathrm{Tor}(F, G)$ satisfies fp_n.

$(C)_n$ If F and G are functors of $fp_n(\mathcal{F}(\mathcal{A};\mathbb{Z}))$, of bounded torsion, and that for each integer $i > 0$, $F_{(i)}$ and $G_{(i)}$ satisfy fp_n, and that T is a bifunctor as in (C) of the previous proposition, then $T \circ (F, G) \in fp_n(\mathcal{F}(\mathcal{A};\mathbb{Z}))$.

$(D)_n$ If F and G are functors of $fp_n(\mathcal{F}(\mathcal{A};\mathbb{Z}))$, of bounded torsion, and that for each integer $i > 0$, $F_{(i)}$ and $G_{(i)}$ satisfy fp_n, then for each $(i, j) \in \mathbb{N}^2$, $F_{(i)} \otimes G_{(j)}$ satisfies fp_n.

We will prove the implications

$$(\mathrm{A})_n \Rightarrow (\mathrm{D})_n \Rightarrow (\mathrm{C})_n \Rightarrow (\mathrm{B})_n \Rightarrow (\mathrm{A})_{n+2}\,.$$

1. $(\mathrm{A})_n \Rightarrow (\mathrm{D})_n$: apply $(\mathrm{A})_n$ to $F_{(i)}$ and $\otimes G_{(j)}$.
2. $(\mathrm{D})_n \Rightarrow (\mathrm{C})_n$: as F and G have bounded torsion, there exists $N \in \mathbb{N}$ such that F and G have values in $\mathbf{Ab}_N$. Then precompose the resolution given by the lemma with (F, G).
3. $(\mathrm{C})_n \Rightarrow (\mathrm{B})_n$: the functor $\mathrm{Tor}_1^{\mathbb{Z}} : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$ satisfies the hypotheses of (C).
4. $(\mathrm{B})_n \Rightarrow (\mathrm{A})_{n+2}$: the hypotheses of $(\mathrm{A})_{n+2}$ are stronger than the ones of $(\mathrm{B})_n$, so $\mathrm{Tor}(F, G)$ satisfies fp_n. As F and G satisfy fp_{n+2}, it follows by the lemma seen at the beginning of this part that $F \otimes G$ satisfies fp_n.

As conditions $(\mathrm{A})_n$, $(\mathrm{B})_n$, $(\mathrm{C})_n$ and $(\mathrm{D})_n$ are obvious for $n < 0$, they hold for all n by induction, whence the proposition.

Proof of the key result on tensor product of fp_n-functors (end)

Notation

Let Tfp_n be the class of functors F of $\mathcal{F}(\mathcal{A};\mathbb{Z})$ such that:

1. F has bounded torsion;
2. F and the $F_{(i)}$'s satisfy fp_n for each $i > 0$.

Corollary

Let F and G be functors of Tfp_n. Then $F \otimes G$ and $\mathrm{Tor}(F, G)$ belong to Tfp_n.

Under the hypotheses of the theorem on tensor products of fp_n-functors, the class $\mathfrak{P}$ is included in Tfp_n (because a noetherian functor has bounded torsion), so this theorem is a special case of the previous corollary.

The end

Thank you for your attention!

Cảm ơn