# Polynomial functors on free groups and on topological spaces after Arone (1): theoretical results

Aurélien DJAMENT

CNRS, LAGA, Villetaneuse, France

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Online talk reporting on (some aspects of) the following recent preprint:

Gregory Arone.
*Polynomial functors from free groups to a stable infinity-category.*
arXiv:2504.04114 [math.AT] (2025)

# A few other references, 1 (topology)

- *S. Galatius*, Stable homology of automorphism groups of free groups. Ann. Math. (2) 173, No. 2, 705–768 (2011).
- *T. G. Goodwillie*, Calculus. I: The first derivative of pseudoisotopy theory. *K*-Theory 4, No. 1, 1–27 (1990).
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- *E. Lindell*, Stable cohomology of $\mathrm{Aut}(F_n)$ with bivariant twisted coefficients. Adv. Math. 480, 46 p. (2025).
- *O. Randal-Williams*, Cohomology of automorphism groups of free groups with twisted coefficients. Sel. Math., New Ser. 24, No. 2, 1453–1478 (2018).

# A few other references, 2 (algebra)


- *A. Djament*, Décomposition de Hodge pour l'homologie stable des groupes d'automorphismes des groupes libres. Compos. Math. 155, No. 9, 1794–1844 (2019).
- *A. Djament* and *C. Vespa*, Sur l'homologie des groupes d'automorphismes des groupes libres à coefficients polynomiaux. Comment. Math. Helv. 90, No. 1, 33–58 (2015).
- *G. Powell* and *C. Vespa*, Higher Hochschild homology and exponential functors. Bull. Soc. Math. Fr. 153, No. 1, 1–141 (2025).
- *O. Randal-Williams* and *N. Wahl*, Homological stability for automorphism groups. Adv. Math. 318, 534–626 (2017).
- *C. Vespa*, Extensions between functors from free groups. Bull. Lond. Math. Soc. 50, No. 3, 401–419 (2018).

# Content of the talks

1. Polynomial functors on free groups: introduction and motivation
   - Basic notions
   - Stable (co)homology of automorphism groups of free groups
   - An example of other use of $\mathcal{F}(\mathbf{gr})$
2. Polynomial functors on **gr**: definition and basic properties
3. Excisive functors (after Goodwillie): statement of the main theoretical result
4. Excisive functors: definition and basic properties
5. Sketch of proof of the main theorem

# Polynomial functors on free groups: introduction and motivation

Let **gr** denote the category of finitely generated free groups.

Let $k$ denote a commutative ground ring.
If $\mathcal{C}$ is an essentially small category, $\mathcal{F}(\mathcal{C}; k)$ denotes the category of functors from $\mathcal{C}$ to $k$-modules. For $k = \mathbb{Z}$, we denote it also only by $\mathcal{F}(\mathcal{C})$.

Into $\mathcal{F}(\mathbf{gr}; k)$, we are interested by the full subcategory $\mathcal{P}ol_d(\mathbf{gr}; k)$ (defined later) of polynomial functors of degree $\leq d$.

A typical example of a polynomial functor of degree $d$ in $\mathcal{P}ol_d(\mathbf{gr})$ is $G \mapsto G_{\mathrm{ab}}^{\otimes d}$, where $G_{\mathrm{ab}}$ is the abelianisation of $G$.

# Why to be interested in it? (1) Stable (co)homology of automorphism groups of free groups

The group $\mathrm{Aut}(F_n)$ was studied from an elementary group-theoretic point of view at least a century ago (Nielsen, Magnus...). But its (co)homology is very hard to compute.

A first important feature is **homological stability**. It was first proven (in the 1990's) for constant coefficients (Hatcher-Vogtmann-Wahl), and later extended to twisted polynomial coefficients by Randal-Williams-Wahl [4].

**Theorem (Randal-Williams-Wahl)**

*Let $X \in \mathcal{P}ol_i(\mathbf{gr}^{\mathrm{op}})$ and $Y \in \mathcal{P}ol_j(\mathbf{gr})$. The canonical map*

$$H_n(\mathrm{Aut}(F_r); X(F_r) \otimes Y(F_r)) \to H_n(\mathrm{Aut}(F_{r+1}); X(F_{r+1}) \otimes Y(F_{r+1}))$$

*is surjective $r \geq 2n+i+j+1$ for and bijective for $r \geq 2n+i+j+3$.*

The value of $H_n(\mathrm{Aut}(F_r); X(F_r) \otimes Y(F_r))$ for $r$ big enough if called stable homology of $\mathrm{Aut}(F_r)$ with coefficients into $X \otimes Y$ and denoted by $H_n^{st}(\mathrm{Aut}(F_\infty); X \otimes Y)$.

**How to compute it?**
(Note: it is still harder to compute $H_n(\mathrm{Aut}(F_r); X(F_r) \otimes Y(F_r))$ for smaller $r$, except if it is really *very* small.)

The answer for constant coefficients is given by Galatius [1] (and Nakaoka's computation of homology of symmetric groups in the early 1960's):

**Theorem (Galatius 2011; very hard)**

*The inclusion $\mathfrak{S}_r \hookrightarrow \mathrm{Aut}(F_r)$ induces stably an isomorphism in homology with constant coefficients.*

# And with polynomial coefficients?

The following result was proven by D.-Vespa in [2] by using homological algebra in the category $\mathcal{F}(\mathbf{gr})$. It was also, independently, essentially established with topological methods by Randal-Williams [6].

**Theorem**

*Let $T$ be a reduced polynomial functor of $\mathcal{F}(\mathbf{gr})$. Then $H_*^{st}(\mathrm{Aut}(F_\infty); T) = 0$.*

Here *reduced* means that $T$ send the trivial group on 0. Every functor of $\mathcal{F}(\mathbf{gr})$ splits uniquely as the direct sum of a constant functor and a reduced one.

The situation is harder and more interesting for polynomial functors of $\mathcal{F}(\mathbf{gr}^{\mathrm{op}})$. I proved in [1] a result whose *rational* form is the following:

**Theorem**

*Let $X$ be a polynomial functor of $\mathcal{F}(\mathbf{gr}^{\mathrm{op}};\mathbb{Q})$. The there is a natural isomorphism*

$$H_n^{st}(\mathrm{Aut}(F_\infty);X) \simeq \bigoplus_{i+j=n} \mathrm{Tor}_i^{\mathbf{gr}}(X,\Lambda^j\circ\mathfrak{a})$$

*where $\Lambda^j$ is the $j$-th exterior power on $\mathbb{Q}$-vector spaces and $\mathfrak{a}\in\mathcal{F}(\mathbf{gr};\mathbb{Q})$ the functor $G\mapsto G_{\mathrm{ab}}\otimes\mathbb{Q}$.*

Here $\mathrm{Tor}_i^{\mathbf{gr}}(X,Y)$ are Tor-groups on the category **gr**, which are dual (as $\mathbb{Q}$-vector spaces) to $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(Y,\mathrm{Hom}_{\mathbb{Q}}(-,\mathbb{Q})\circ X)$.

Over the integers, the statement is more technical:

## Theorem

*Let $X$ be a polynomial functor of $\mathcal{F}(\mathbf{gr}^{\mathrm{op}})$. The there is a natural isomorphism*

$$H_n^{st}(\mathrm{Aut}(F_\infty); X) \simeq \bigoplus_{i+j+t=n} H_t\Big(\mathfrak{S}_\infty; \mathbb{H}_i(\mathfrak{S}_j; X \overset{\mathbf{L}}{\underset{\mathbf{gr}}{\otimes}} \mathfrak{a}_{\mathrm{sgn}}^{\otimes j})\Big)$$

*where $\mathbb{H}$ denotes hyperhomology, the infinite symmetric group $\mathfrak{S}_\infty$ acts trivially on the right hand side, the index $\mathrm{sgn}$ means that the canonical action of the symmetric group $\mathfrak{S}_j$ on $\mathfrak{a}^{\otimes j}$ is twisted by signature, and $\mathfrak{a}$ denotes the abelianisation functor.*

At least over $\mathbb{Q}$, these stable homology groups can be computed for a lot of polynomial functors of $\mathcal{F}(\mathbf{gr}^{\mathrm{op}})$, especially thank to Vespa's results [5]. But not all computations are known.

Here also, independent methods from algebraic topology can lead to essentially equivalent results (over $\mathbb{Q}$ only), thanks to Randal-Williams [6].

And what's about $H_n^{st}(\mathrm{Aut}(F_\infty); X \otimes Y)$ when $X \in \mathcal{F}(\mathbf{gr}^{\mathrm{op}})$ and $Y \in \mathcal{F}(\mathbf{gr})$ are non-constant polynomial functors?

I gave in [1] a conjecture relating this to homological algebra on the category $\mathcal{F}(\mathbf{gr})$, allowing to make rational computations for 'reasonable' polynomial functors $X$ and $Y$.

The rational form of this conjecture has been recently proved by topological methods (related to Randal-Williams' ones [6]) by Lindell [5].

# Why to be interested in $\mathcal{P}ol_d(\mathbf{gr})$? (2)

Other questions related to algebraic topology make polynomial functors on **gr** 'naturally' appear.

In [3], Powell and Vespa study higher Hochschild homology (introduced by Pirashvili, *Ann. Sci. ENS* 2000) associated to wedges of circles with coefficients into square-zero extensions of field of characteristic 0 by considering it as a functor on free groups (through the classifying space functor).
Their article studies also from the point of view of $\mathcal{P}ol_d(\mathbf{gr})$ the question whether a linear representation of $\mathrm{Aut}(F_r)$ factors, or not, through the canonical projection onto $\mathrm{Out}(F_r)$.

# Polynomial functors on **gr**: definition

## Definition (Eilenberg-MacLane)

Let $(\mathcal{C}, \star, 0)$ be a symmetric monoidal category whose unit object 0 is a null object (i.e. both initial and terminal), $\mathcal{E}$ an abelian category and $F : \mathcal{C} \to \mathcal{E}$ a functor. The $n$-th **cross-effect** of $F$ is the functor

$$cr_n(F) : \mathcal{C}^n \to \mathcal{E} \quad (t_1, \dots, t_n) \mapsto \mathrm{Ker}\left( F\left(\bigstar_{i=1}^n t_i\right) \to \bigoplus_{j=1}^n F\left(\bigstar_{i \neq j}^n t_i\right)\right).$$

One says that $F$ is **polynomial** of degree $\leq d$ if $cr_{d+1}(F) = 0$.

# General basic properties

## Notation

Let $(\mathcal{C}, *, 0)$ be an essentially small symmetric monoidal category whose unit object 0 is a null object and $\mathcal{E}$ an abelian category. We denote by $\mathcal{P}ol_d(\mathcal{C}, \mathcal{E})$ the full subcategory of $\mathbf{Fun}(\mathcal{C}, \mathcal{E})$ consisting of polynomial functors of degree $\leq d$. If $\mathcal{E} = \mathbf{Ab}$ (resp. $k$-$\mathbf{Mod}$), we denote it simply by $\mathcal{P}ol_d(\mathcal{C})$ (resp. $\mathcal{P}ol_d(\mathcal{C}; k)$).

The subcategory $\mathcal{P}ol_d(\mathcal{C}, \mathcal{E})$ of $\mathbf{Fun}(\mathcal{C}, \mathcal{E})$ is stable under subobjects, quotients, arbitrary limits and colimits. In $\mathcal{F}(\mathcal{C}; k)$, the tensor product (over $k$, pointwise) of a functor of $\mathcal{P}ol_i(\mathcal{C})$ and a functor of $\mathcal{P}ol_j(\mathcal{C})$ belongs to $\mathcal{P}ol_{i+j}(\mathcal{C})$.

# Polynomial functors on gr and abelianisation

All functors of $\mathcal{P}ol_1(\mathbf{gr}, k)$ factorise through abelianisation and are isomorphic to $G \mapsto U \oplus (G_{\mathrm{ab}} \otimes_{\mathbb{Z}} V)$ for some $k$-modules $U$ and $V$. More generally:

**Proposition**

*Each functor $F$ of $\mathcal{P}ol_d(\mathbf{gr}, \mathcal{E})$ (where $\mathcal{E}$ is an abelian category) admits a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_d = F$$

*such that all subquotients $F_i/F_{i-1}$ factor through abelianisation.*

For $d \geq 1$, the $d$-th **Passi functor** is the functor of $\mathcal{F}(\mathbf{gr})$ sending $G$ to the augmentation ideal of its group ring $\mathbb{Z}[G]$ quotiented out by its $(d+1)$ power. It belongs to $\mathcal{P}ol_d(\mathbf{gr})$ but, for $d \geq 2$, it doesn't factor through abelianisation.

# Homotopical categorical setting for excisive functors

Let $\mathcal{C}$ denote the category of pointed spaces — where *space*=topological space or simplicial set — or a 'nice' full subcategory of it, and $\mathcal{D}$ denote the category of (unbounded) chain complexes of abelian groups (or any other nice abelian category). We are now interested with functors from $\mathcal{C}$ to $\mathcal{D}$ which preserve filtered homotopy colimits and weak equivalences.

Arone [1] uses a more general framework, where $\mathcal{C}$ is a (nice) pointed $\infty$-category and $\mathcal{D}$ a (nice) stable $\infty$-category, but applies it only in the previous setting.

What is important for the formal setting is to have notions of weak equivalences, homotopy limits and homotopy colimits which behave in the usual way in $\mathcal{C}$ and $\mathcal{D}$, that $\mathcal{C}$ is pointed with coproducts, and that $\mathcal{D}$ is additive.

# Excisive functors (after Goodwillie)

Goodwillie introduced and studied in [2, 3, 4] the notion of *n-excisive* functors from $\mathcal{C}$ to $\mathcal{D}$ (definition reminder soon). This is a homotopical variation around the notion of polynomial functor of degree $\leq n$ in the classical (Eilenberg-MacLane) sense.

One denotes by $\mathrm{Exc}_n(\mathcal{C}, \mathcal{D})$ the category of functors (preserving filtered homotopy colimits and weak equivalences) $\mathcal{C} \to \mathcal{D}$ which are $n$-excisive.

# The main theoretical theorem of Arone's work

Let $\mathbf{S}_*$ denote the category of pointed spaces. The classifying space construction defines a functor $\mathbf{gr} \to \mathbf{S}_*$. Restriction along it induces a functor $\mathrm{Exc}_n(\mathbf{S}_*, \mathcal{D}) \to \mathcal{P}ol_n(\mathbf{gr}, \mathcal{D})$.

**Remark**

Here $\mathcal{P}ol_n(\mathbf{gr}, \mathcal{D})$ consists of functors $F : \mathbf{gr} \to \mathcal{D}$ such that $cr_{n+1}(F) \simeq 0$, where $\simeq$ means *weak* equivalence.

**Theorem (Arone)**

*This restriction functor is a weak equivalence*

$$\mathrm{Exc}_n(\mathbf{S}_*, \mathcal{D}) \xrightarrow{\simeq} \mathcal{P}ol_n(\mathbf{gr}, \mathcal{D}) .$$

# Corollary: towards Ext-computations on gr

## Corollary (Arone)

*Let $F$ be a functor of $\mathcal{P}ol_n(\mathbf{gr})$. Then there exists an n-excisive functor $\hat{F} : \mathbf{S}_* \to \mathbf{Ch}(\mathbf{Ab})$, unique up to (weak) equivalence, such that $\hat{F}(X) \simeq F(\pi_1(X))$ naturally in $X$ when $X$ is a (finite) wedge of circles. Furthermore, if $G$ is another polynomial functor of $\mathcal{F}(\mathbf{gr})$, then there is a natural isomorphism of graded abelian groups*

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(F, G) \simeq \pi_{-*}(\underline{\mathrm{Hom}}(\hat{F}, \hat{G}))$$

*where $\underline{\mathrm{Hom}}$ denotes the enriched Hom-functor (with values into $\mathbf{Ch}(\mathbf{Ab})$) and $\pi_*$ denotes the homology of a chain complex of abelian groups.*

# Cubical diagrams

To define excisive functors, we need to introduce cubical diagrams.

## Definition

Let $n \geq 0$ be an integer and $\mathcal{C}$ a category. An *n*-**cubical diagram** of $\mathcal{C}$ is a functor from the poset $\mathcal{P}(n)$ of subsets of $\mathbf{n} := \{1, \ldots, n\}$ (ordered by inclusion), seen as a small category, to $\mathcal{C}$.

## Example

Assume that $\mathcal{C}$ has finite coproducts. If $x_0, \ldots, x_n$ are objects of $\mathcal{C}$, then one has an *n*-cubical diagram given by $E \subset \mathbf{n} \mapsto x_0 \sqcup \bigsqcup_{i \in E} x_i$, with structure morphisms given by the universal property of coproducts.

## Definition

Let $\mathcal{C}$ be a category with finite limits and colimits (resp. a model category, or $\infty$-category...) and $\chi : \mathcal{P}(n) \to \mathcal{C}$ an $n$-cubical diagram. One says that $\chi$ is:

- cartesian (resp. homotopy cartesian) if the natural map

$$\chi(\varnothing) \to (\mathrm{ho})\lim_{\varnothing \neq U \subset \mathbf{n}} \chi(U)$$

  is an isomorphism (resp. a weak equivalence).
- cocartesian (resp. homotopy cocartesian) if the natural map

$$(\mathrm{ho})\mathrm{colim}_{U \subsetneq \mathbf{n}} \chi(U) \to \chi(\mathbf{n})$$

  is an isomorphism (resp. a weak equivalence).

### Remark

If $\mathcal{C}$ is abelian (resp. a *stable* $\infty$-category, as spectra, or $\mathbf{Ch}(\mathcal{E})$ for $\mathcal{E}$ an abelian category), then a cubical diagram of $\mathcal{C}$ is (homotopy) cartesian if and only if it is (homotopy) cocartesian.

## Definition

Let $\mathcal{C}$ be a model category (or $\infty$-category...) and $\chi : \mathcal{P}(n) \to \mathcal{C}$ an $n$-cubical diagram. One says that $\chi$ is *strongly homotopy cocartesian* if it is equivalent to the (derived) left Kan extension of its restriction to subsets of cardinality $\leq 1$ of $\mathbf{n}$, i.e. for every $U \subset \mathbf{n}$ the canonical arrow

$$\operatorname*{hocolim}_{V \subset U, \mathrm{Card}(V) \leq 1} \chi(V) \to \chi(U)$$

is a weak equivalence.

Every 1-cubical diagram is strongly homotopy cocartesian; for a 2-cubical diagram strongly homotopy cocartesian is the same as homotopy cocartesian; for $n > 2$ any $n$-cubical homotopy strongly cocartesian diagram is homotopy cocartesian, but the converse is wrong.

# Definition of excisive functors

## Definition

Let $n \geq 0$ be an integer, $\mathcal{C}$ and $\mathcal{D}$ be 'nice' model category (or $\infty$-category). A functor $F : \mathcal{C} \to \mathcal{D}$ preserving weak equivalences and filtered homotopy colimits is said to be *n-excisive* if it sends every strongly homotopy cocartesian $(n+1)$-cubical diagram of $\mathcal{C}$ to a homotopy cartesian cubical diagram of $\mathcal{D}$.

## Remark

If $\mathcal{C}$ is a pointed category with finite coproducts and $\mathcal{E}$ an abelian category, a functor $\mathcal{C} \to \mathcal{E}$ is polynomial of degree $\leq n$ (in the sense of Eilenberg-MacLane) if and only if it sends every $(n+1)$-cubical diagram of the shape described in the previous example to a cartesian cubical diagram (which is the same, here, as a cocartesian diagram).

# Steps of the proof of the main theorem

The equivalence of Arone's theorem decomposes into three equivalences:

$$\mathrm{Exc}_n(\mathbf{S}_*, \mathcal{D}) \underset{(1)}{\xrightarrow{\simeq}} \mathrm{Exc}_n(\mathbf{S}_*^c, \mathcal{D}) \underset{(2)}{\xrightarrow{\simeq}} \mathrm{Exc}_n(s\mathbf{Grp}, \mathcal{D}) \underset{(3)}{\xrightarrow{\simeq}} \mathcal{P}ol_n(\mathbf{gr}, \mathcal{D})$$

Here $\mathcal{D}$ is $\mathbf{Ch}(\mathbf{Ab})$, or more generally a (nice) model category or a (nice) stable $\infty$-category; $\mathbf{S}_*$ is the category of pointed spaces, $\mathbf{S}_*^c$ its full subcategory of *connected* pointed spaces, and $s\mathbf{Grp}$ the category of simplicial groups.

# The equivalence (1)

This equivalence $\mathrm{Exc}_n(\mathbf{S}_*, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Exc}_n(\mathbf{S}_*^c, \mathcal{D})$ is induced by restriction from pointed spaces to connected pointed spaces.

But notice that this equivalence is specific to excisive functors: the restriction functor $\mathbf{Fun}(\mathbf{S}_*, \mathcal{D}) \to \mathbf{Fun}(\mathbf{S}_*^c, \mathcal{D})$ is *not* an equivalence.

The proof of (1) uses a construction due to Goodwillie.
Let $*$ denote the *join* of spaces. When one of the spaces is pointed, the join inherits a base point. One can restrict the join into a functor $\mathbf{S}_* \times \mathbf{FI} \to \mathbf{S}_*$, where **FI** is the category of finite sets (seen as discrete spaces) with injections.

## Definition (Goodwillie)

Let $F : \mathbf{S}_* \to \mathcal{D}$ be a functor and $n \in \mathbb{N}$. One defines a functor $T_n F : \mathbf{S}_* \to \mathcal{D}$ by

$$(T_n F)(X) := \operatorname*{holim}_{\varnothing \neq U \subset \mathbf{n+1}} F(X * U) .$$

One has a natural map $F \to T_n F$.

## Lemma

*If $F$ is $n$-excisive, then this map is an equivalence.*

## Proof.

It follows from the observation that for any pointed space $X$ and any $i \in \mathbb{N}$, the $i$-cubical diagram mapping a subset $U$ of $\mathbf{i}$ on $X * U$ is strongly homotopy cocartesian. □

**Lemma**

*For any non-empty finite set $U$ and any space $X$, the space $X * U$ is connected.*

The proof of the equivalence (1) is not very difficult from the previous lemmas.

A (weak) pseudo-inverse to the restriction functor $\mathrm{Exc}_n(\mathbf{S}_*, \mathcal{D}) \to \mathrm{Exc}_n(\mathbf{S}_*^c, \mathcal{D})$ is given by the restriction to $n$-excisive functors of the right (derived) Kan extension $\mathbf{Fun}(\mathbf{S}_*^c, \mathcal{D}) \to \mathbf{Fun}(\mathbf{S}_*, \mathcal{D})$.

# The equivalence (2)

The equivalence $\mathrm{Exc}_n(\mathbf{S}_*^c, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Exc}_n(s\mathbf{Grp}, \mathcal{D})$ is the restriction to $\mathrm{Exc}_n(\mathbf{S}_*^c, \mathcal{D})$ of an equivalence $\mathbf{Fun}(\mathbf{S}_*^c, \mathcal{D}) \xrightarrow{\simeq} \mathbf{Fun}(s\mathbf{Grp}, \mathcal{D})$. This equivalence is obtained by precomposition by the classifying space functor $\mathrm{B} : s\mathbf{Grp} \to \mathbf{S}_*^c$; the Kan loop group functor $\Omega : \mathbf{S}_*^c \to \mathbf{Grp}$ is a weak inverse of it.

# The equivalence (3)

The equivalence $\mathrm{Exc}_n(s\mathbf{Grp}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{P}ol_n(\mathbf{gr}, \mathcal{D})$ is induced by restriction from simplicial groups to the ones which are constant and free of finite rank.

This comes from the propositions below.

**Proposition**

*Every excisive functor commutes with sifted colimits.*

**Proposition**

*The category $s\mathbf{Grp}$ is the sifted completion of the category $\mathbf{gr}$.*

(This means that the inclusion $\mathbf{gr} \hookrightarrow s\mathbf{Grp}$ induces an equivalence between functors from $s\mathbf{Grp}$ that commute to sifted colimits and functors from $\mathbf{gr}$.)

# Remark on the composite equivalence

The equivalence $\mathrm{Exc}_n(\mathbf{S}_*, \mathcal{D}) \xrightarrow{\simeq} \mathcal{P}ol_n(\mathbf{gr}, \mathcal{D})$ that we get is induced by precomposition by the classifying space functor $\mathbf{gr} \to \mathbf{S}_*$. But there is no easy 'formula' for a weak inverse, because one uses the *right* derived Kan extension for the inverse at step (1) and *left* derived Kan extension for the inverse at step (3).

# (Temporary) end

Thank you for your attention!

Cám on vì su quan tâm cùa ban !