Polynomial functors on free groups and on topological spaces after Arone (2): Ext-computations

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Second (and final) talk of an online mini-course to Vietnam Institute for Advanced Study in Mathematics (VIASM).

Online talk reporting on (some aspects of) the following recent preprint:

Gregory Arone.
Polynomial functors from free groups to a stable infinity-category.
arXiv:2504.04114 [math.AT] (2025)

For the third part of this talk, we will follow:

C. Vespa, Extensions between functors from free groups. Bull. Lond. Math. Soc. 50, No. 3, 401–419 (2018).

Content of the talks

1. Computation of $\hat{F}$ for some polynomial functors F on $\mathbf{gr}$
2. Computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ after Arone
3. Computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ after Vespa

Reminder of the previous talk

To any $F \in \mathcal{P}ol_n(\mathbf{gr})$ (that is, a polynomial functor $\mathbf{gr} \to \mathbf{Ab}$ of degree $\leq n$, $\mathbf{gr}$ being the category of finitely-generated free groups) Arone associates functorially a functor $\hat{F} \in \mathrm{Exc}_n(\mathbf{S}_*, \mathbf{Ch}(\mathbf{Ab}))$ (that is, an n-excisive functor $\mathbf{S}_* \to \mathbf{Ch}(\mathbf{Ab})$, $\mathbf{S}_*$ being the category of pointed spaces), unique up to weak equivalence, in such a way that

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(F, G) \simeq \pi_{-*}(\underline{\mathrm{Hom}}(\hat{F}, \hat{G}))$$

(where G is another polynomial functor on $\mathbf{gr}$).

Moreover, $\hat{F}$ is characterised by the natural equivalence $\hat{F}(\mathrm{B}G) \simeq F(G)$ for a finitely generated free group G (up to the canonical inclusion $\mathbf{Ab} \hookrightarrow \mathbf{Ch}(\mathbf{Ab})$).

Functors factorising through abelianisation

For a pointed space X, we denote by $\tilde{C}_*(X)$ its reduced chain complex. If 'space' means *simplicial set*, we can so take $\tilde{C}_n(X) = \mathbb{Z}[X_n]/\mathbb{Z}$ (where the factor $\mathbb{Z}$ comes from the base point) with the usual simplicial differential.

We denote by Σ the suspension functor of $\mathbf{Ch}(\mathbf{Ab})$ and by Σ^i its i-th iteration, for $i \in \mathbb{Z}$.

Remind that $\mathfrak{a} \in \mathcal{F}(\mathbf{gr})$ denotes the abelianisation functor.

Proposition (Arone's Lemma 9.1)

For any $n \in \mathbb{N}$, there is a natural equivalence

$$\widehat{\mathfrak{a}^{\otimes n}}(X) \simeq \Sigma^{-n}\tilde{C}_*(X^{\wedge n}) .$$

Proof.

The endofunctor $X \mapsto X^{\wedge n}$ of $\mathbf{S}_*$ maps strongly homotopy cocartesian n-cubical diagrams to homotopy cocartesian diagrams. The functor $\tilde{C}_*$ preserves homotopy cocartesian diagrams, and homotopy cocartesian diagrams are the same as homotopy cartesian diagrams in $\mathbf{Ch}(\mathbf{Ab})$. As Σ^{-n} is a self-equivalence of this category, this implies that the functor $X \mapsto \Sigma^{-n}\tilde{C}_*(X^{\wedge n})$ is n-excisive.
So, it is enough to prove a natural equivalence $G_{\mathrm{ab}}^{\otimes n} \simeq \Sigma^{-n}\tilde{C}_*(\mathrm{B}G^{\wedge n})$ for $G \in \mathbf{gr}$. It follows from the fact that $\tilde{C}_*(\mathrm{B}G^{\wedge n}) \simeq \tilde{C}_*(\mathrm{B}G)^{\otimes n}$ (Künneth) and that $\tilde{C}_*(\mathrm{B}G)$ has at most a nonzero homology group, in degree 1, naturally isomorphic to G_{ab}. □

Remark

The equivalence of the previous proposition is $\mathfrak{S}_n$-equivariant.

Proposition (Arone's Proposition 9.4 (1))

For any $n \in \mathbb{N}$, there is a natural equivalence

$$\widehat{\Lambda^n \circ \mathfrak{a}}(X) \simeq \Sigma^{-n} \tilde{C}_*(X^{\wedge n}_{\mathfrak{S}_n}) ,$$

where Λ^n denotes the n-the exterior power.

Note that the index $\mathfrak{S}_n$ indicates the *naive* coinvariants under the action of the symmetric group, and not the homotopy coinvariants.

Rationally, both are the same, and it follows from the previous proposition and remark.

Arone gives, in his Theorem 9.7, a general recipe to compute $\hat{F}$ for a polynomial functor $F \in \mathcal{F}(\mathbf{gr})$ factorising through the abelianisation. Precisely, for a polynomial functor T from finitely-generated free *abelian* groups to abelian groups, Arones proves a natural equivalence

$$\widehat{T \circ \mathfrak{a}}(X) \simeq T\big(\Omega \tilde{C}_*(X)\big)$$

but we have to be careful about what means the 'loop functor' Ω here, what is not completely clear for me... (And T on the right hand side means a suitable pointwise extension of T.)

An other example: Passi functors

[1, Proposition 10.1] describes $\widehat{\mathrm{Pa}}_n$, where $\mathrm{Pa}_n \in \mathcal{P}ol_n(\mathbf{gr})$ denotes the n-th Passi functor, as $P_n(\tilde{C}_* \circ \Omega)$, where Ω is the loop space functor on $\mathbf{S}_*$ and P_n denotes the n-th excisive approximation, that is the left adjoint of the inclusion $\mathrm{Exc}_n(\mathbf{S}_*, \mathbf{Ch}(\mathbf{Ab}))$ into the category of all functors preserving filtered homotopy colimits and weak equivalences, which was constructed by Goodwillie.

Cross-effects in the sense of Goodwillie

Definition (Arone's Definition 11.1, after Goodwillie)

Let $F : \mathbf{S}_* \to \mathbf{Ch}(\mathbf{Ab})$ be a functor and $n \in \mathbb{N}$. The n-th cross-effect functor of F is the functor $cr_n(F) : \mathbf{S}_*^n \to \mathbf{Ch}(\mathbf{Ab})$ mapping $(X_1, \dots, X_n)$ on the homotopy fiber of the functor defined on non-empty subsets of $\mathbf{n}$ that sends $U \subset \mathbf{n}$ on $F\left(\bigvee_{i \notin U} X_i\right)$ (inclusions being sent on obvious collapsing maps).

(One can replace $\mathbf{Ch}(\mathbf{Ab})$ at the target by any stable model category or stable ∞-category.)

So, F belongs to $\mathrm{Exc}_n(\mathbf{S}_*, \mathbf{Ch}(\mathbf{Ab}))$ if and only if $cr_{n+1}(F) \simeq 0$.

Remark

For a functor T from an additive category to an abelian one, one has natural decompositions

$$T\left(\bigoplus_{i=1}^{n} a_i\right) \simeq \bigoplus_{I\subset \mathbf{n}} cr_I(T)(a|_I)$$

where $cr_I(T)(a|_I) := cr_r(T)(a_{i_1}, \dots, a_{i_r})$ if $I = \{i_1, \dots, i_r\}$ with $i_1 < \dots < i_r$.
Similarly, for a functor $F : \mathbf{S}_* \to \mathbf{Ch}(\mathbf{Ab})$, one has natural decompositions

$$F\left(\bigvee_{i=1}^{n} X_i\right) \simeq \bigoplus_{I\subset \mathbf{n}} cr_I(F)(X|_I)\ .$$

Computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$: preparation

Arone's computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ relies on the more general following result.

Proposition (Arone's Proposition 11.2)

Let F be a polynomial functor of $\mathcal{F}(\mathbf{gr})$. Then we have a natural isomorphism of graded abelian groups

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, F) \simeq \pi_{-*-n} cr_n(\hat{F})(S^0, \dots, S^0)$$

($\mathfrak{S}_n$-equivariant if we twist the obvious action on the right hand side by the sign representation).

The proof relies, by a formal adjunction argument, on the following result.

Proposition (Arone's Lemma 11.3))

Working in the category of functors (preserving weak equivalences and filtered homotopy colimits) from $\mathbf{S}_*$ *to spectra, there is a natural weak equivalence*

$$\underline{\mathrm{Hom}}(X \mapsto \Sigma^\infty(X^{\wedge n}), T) \simeq cr_n(T)(S^0, \dots, S^0)$$

in the category of spectra.

The proof relies on the two following results:

- $\Sigma^\infty X^{\wedge n}$ is naturally the homotopy cofiber of the functor defined on strict subsets of $\mathbf{n}$ mapping U on $\Sigma^\infty X^U$;
- as $X^n \simeq \underline{\mathrm{Hom}}(\mathbf{n}_+, X)$ (where $\mathbf{n}_+$ is the pointed space obtained from the discrete space $\mathbf{n}$ by adding an external base point), a variation around Yoneda's Lemma gives

$$\underline{\mathrm{Hom}}(X \mapsto \Sigma^\infty(X^n), T) \simeq T(\mathbf{n}_+) \simeq T(S^0 \vee \cdots \vee S^0) .$$

Theorem (Vespa, recovered by Arone's Corollary 11.4 (1))

The abelian group $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ *is free of rank the number* $\mathrm{Surj}(m, n)$ *of surjections* $\mathbf{m} \twoheadrightarrow \mathbf{n}$ *if* $i = m - n$ *and zero for* $i \neq m - n$.

Proof.

From the previous results we get isomorphisms

$$\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m}) \simeq \pi_{-*-n}(cr_n(\widehat{a^{\otimes m}})(S^0, \dots, S^0)) \simeq$$

$$\pi_{-*-n}(cr_n(X \mapsto \Omega^m \tilde{C}_*(X^{\wedge m}))(S^0, \dots, S^0)) \simeq$$

$$\pi_{m-n-*}(cr_n(X \mapsto \tilde{C}_*(X^{\wedge m}))(S^0, \dots, S^0)).$$

We note that $\tilde{C}_*((S^0 \vee \dots \vee S^0)^{\wedge m}))$ is quasi-isomorphic to a free abelian group of rank n^m concentrated in degree 0, and in it one can identify by elementary combinatorial arguments the 'piece' corresponding to the cross-effect as a free abelian group of rank $\mathrm{Surj}(m, n)$. □

Another Ext-computation by Arone

Arone gives also the following example of Ext-computation relying on his general results:

Proposition (Arone's Corollary 11.14)

Let i and n be nonnegative integers. Then $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathrm{Pa}_n)$ is isomorphic to $\mathbb{Z}$ if both following conditions hold:
n is even and $i = n - 1$
and is zero in all other cases.

(The proof is not immediate and involves combinatorial and topological arguments from the computation of $\hat{\mathfrak{a}}$ and $\widehat{\mathrm{Pa}_n}$.)

Vespa's original method for this computation — input

As the homology of a free group is concentrated in degrees 0 and 1, by truncating the degree 0 and shifting by 1 the bar resolution, one gets a projective resolution in $\mathcal{F}(\mathbf{gr})$ of the abelianisation functor $\mathfrak{a}$ as follows:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1$$

where $P_i(G) := \mathbb{Z}[G^i]$, that is $P_i = \mathbb{Z}[\mathbf{gr}(\mathbb{Z}^{*i}, -)]$, which is projective thanks to the Yoneda lemma.
Variation: one may use the *normalised* bar construction, getting another projective resolution of $\mathfrak{a}$ in $\mathcal{F}(\mathbf{gr})$:

$$\cdots \to \bar{P}^{\otimes n} \to \bar{P}^{\otimes (n-1)} \to \cdots \to \bar{P}$$

where $\bar{P}(G)$ is the augmentation ideal of $\mathbb{Z}[G]$.

Computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F)$

So, for any $F \in \mathcal{F}(\mathbf{gr})$, $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F)$ is naturally isomorphic to the cohomology of a complex

$$F(\mathbb{Z}) \to F(\mathbb{Z}^{*2}) \to \cdots \to F(\mathbb{Z}^{*n}) \to F(\mathbb{Z}^{*(n+1)}) \to \cdots$$

and of a complex

$$cr_1(F)(\mathbb{Z}) \to cr_2(F)(\mathbb{Z}, \mathbb{Z}) \to \cdots \to cr_n(F)(\mathbb{Z}, \ldots, \mathbb{Z}) \to \cdots \ .$$

It follows that:

Proposition

For $F \in \mathcal{P}ol_d(\mathbf{gr})$ and $n \geq d$, $\mathrm{Ext}^n_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F) = 0$.

In particular:

Proposition

The graded group $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a})$ is isomorphic to $\mathbb{Z}$ concentrated in degree 0.

(Compare to the situation of free *abelian* groups at the source, where the similar computation of Ext between additive functors is much richer and harder.)

Another example of easy computation

For $F = \Lambda^n(\mathfrak{a})$ (n-th exterior power of the abelianisation), $cr_i(F)(\mathbb{Z}, \dots, \mathbb{Z})$ is zero except if $i = n$, so we get:

Proposition

Let i and n be nonnegative integers. The group $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \Lambda^n(\mathfrak{a}))$ is isomorphic to $\mathbb{Z}$ if $i = n - 1$ and is zero else.

A Künneth formula for $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, -)$

Another consequence of the bar resolution of $\mathfrak{a}$ is the following.

Proposition

Let $F, G \in \mathcal{F}(\mathbf{gr})$. Assume that the values of F and all groups $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F)$ are torsion-free. Assume moreover that F and G are reduced. Then one has a natural isomorphism

$$\mathrm{Ext}^n_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F \otimes G) \simeq \bigoplus_{i+j=n-1} \mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F) \otimes \mathrm{Ext}^j_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, G)$$

for each integer $n \geq 0$.

Computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes m})$

From the previous results we deduce:

Proposition (Vespa)

For any $m, i \in \mathbb{N}$, the group $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes m})$ is isomorphic to $\mathbb{Z}$ if $i = m - 1$ and is zero else.

Another Künneth formula in functor cohomology

Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ be essentially small categories and $F, G \in \mathcal{F}(\mathcal{C})$, $T, U \in \mathcal{F}(\mathcal{D})$. One assume that F and T have projective resolutions whose terms are all finitely-generated and that the values of G and all groups $\mathrm{Ext}^i_{\mathcal{F}(\mathcal{C})}(F, G)$ are torsion-free. Then one has a natural isomorphism of graded abelian groups

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}\times\mathcal{D})}(F \boxtimes T, G \boxtimes U) \simeq \mathrm{Ext}^*_{\mathcal{F}(\mathcal{C})}(F, G) \otimes \mathrm{Ext}^*_{\mathcal{F}(\mathcal{D})}(T, U).$$

Here $\boxtimes$ denotes the external tensor product: $F \boxtimes T$ is the functor of $\mathcal{F}(\mathcal{C} \times \mathcal{D})$ defined by $(F \boxtimes T)(x, y) := F(x) \otimes T(y)$.

Corollary

Let $F_1, \dots, F_n$ be functors of $\mathcal{F}(\mathbf{gr})$. One assume that the values of the F_i's and all groups $\mathrm{Ext}^r_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F_i)$ are torsion-free. Then one has a natural isomorphism of graded abelian groups

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}^n)}(\mathfrak{a}^{\boxtimes n}, F_1 \boxtimes \cdots \boxtimes F_n) \simeq \bigotimes_{i=1}^{n} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F_i).$$

Adjunction between coproduct and diagonal

For $n \in \mathbb{N}$, let $\delta_n : \mathbf{gr} \to \mathbf{gr}^n$ denote the n-th iterated diagonal functor and $*_n : \mathbf{gr}^n \to \mathbf{gr}$ denote the n-th iterated free product functor. As the free product is a categorical coproduct, $*_n$ is left-adjoint to δ_n. It formally implies that the precomposition by $*_n$ is *right*-adjoint to the precomposition δ_n. As each precomposition functor is exact, this adjunction propagates to Ext-groups: for any functors $F \in \mathcal{F}(\mathbf{gr})$ and $X \in \mathcal{F}(\mathbf{gr}^n)$, one has a natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(X \circ \delta_n, F) \simeq \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}^n)}(X, F \circ *_n)$$

of graded abelian groups.

Original (algebraic) proof of Vespa's Theorem for $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$

As $\mathfrak{a}^{\otimes n} \simeq \mathfrak{a}^{\boxtimes n} \circ \delta_n$, the previous Ext-adjunction specialises as

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m}) \simeq \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}^n)}(\mathfrak{a}^{\boxtimes n}, \mathfrak{a}^{\otimes m} \circ *_n)$$

Furthermore, one has an isomorphism

$$\mathfrak{a}^{\otimes m} \circ *_n \simeq \bigoplus_{i_1+\cdots+i_n=m} (\mathfrak{a}^{\otimes i_1} \boxtimes \cdots \boxtimes \mathfrak{a}^{\otimes i_n}) \uparrow^{\mathfrak{S}_m}_{\mathfrak{S}_{i_1}\times\cdots\times\mathfrak{S}_{i_n}}$$

as functors with a $\mathfrak{S}_m$-action.

So, using Künneth formula and the computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes m})$, one gets $\mathfrak{S}_m$-equivariant isomorphisms

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m}) \simeq \bigoplus_{i_1+\cdots+i_n=m} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}^n)}(\mathfrak{a}^{\boxtimes n}, \mathfrak{a}^{\otimes i_1} \boxtimes \cdots \boxtimes \mathfrak{a}^{\otimes i_n}) \uparrow^{\mathfrak{S}_m}_{\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_n}}$$

$$\simeq \bigoplus_{i_1+\cdots+i_n=m} \left(\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes i_1}) \otimes \cdots \otimes \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes i_n})\right) \uparrow^{\mathfrak{S}_m}_{\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_n}}$$

$$\simeq \bigoplus_{\substack{i_1+\cdots+i_n=m \\ i_1,\dots,i_n>0}} \Sigma^{m-n}\mathbb{Z} \uparrow^{\mathfrak{S}_m}_{\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_n}} .$$

(Here, $\mathbb{Z}$ is seen as a trivial representation of $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_n}$, but one has to be careful with signs.)

So, $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ is zero if $i \neq m - n$, and is a free abelian group of rank

$$\sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\ldots,i_n>0}} \frac{m!}{i_1!\ldots i_n!} ,$$

which is exactly the cardinality of the set of surjections $\mathbf{m} \twoheadrightarrow \mathbf{n}$ (count it with orbits under the action of $\mathfrak{S}_m$), what finishes Vespa's proof.

Additional remarks about $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$

The family of graded abelian groups $(\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m}))_{(n,m)\in\mathbb{N}^2}$ carries a rich structure: it is a PROP (so, $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ has commuting actions of $\mathfrak{S}_n$ and $\mathfrak{S}_m$, we have Yoneda compositions and monoidal products, which satisfy several compatibility properties).

This structure was completely determined by Vespa [1].

Arone recovers the actions of symmetric groups, but doesn't say if his methods permit to recover the compositions.

Final (open) questions

What is the conceptual relation between Vespa's algebraic approach and Arone's topological proof?

Can one get the new $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}$ between polynomial functors made by Arone in [1] by going on Vespa's methods?

The end

Thank you for your attention!

Cám on vì su quan tâm cùa ban !