Stable homology of automorphism groups of free groups and functor homology (II)

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This talk reports on the preprint [3].

Warning: some of the notations used in these notes do not exactly coincide with the notations of the corresponding preprint. Morover, these notes do not faithfully coincide with the oral talk.

1 Main results

In the following statement, **gr** denotes the category of free groups of finite rank, $\mathbf{Vec}_\mathbb{Q}$ the category of $\mathbb{Q}$-vector spaces, $\mathfrak{a} : \mathbf{gr} \to \mathbf{Vec}_\mathbb{Q}$ the composition of abelianisation and rationalisation ($\mathfrak{a}(G) = G_{ab} \otimes \mathbb{Q}$) and Λ^j the j-th exterior power. As usual, F_r is a free group of rank r; **gr-Mod** is the category of functors from **gr** to $\mathbf{Vec}_\mathbb{Q}$.

Theorem 1. *Let $F : \mathbf{gr} \to \mathbf{Vec}_\mathbb{Q}$ be a polynomial functor of degree d and n, r non-negative integers such that $r \geq 2(n + d) + 3$. Then there is a natural isomorphism*

$$H^n(\mathrm{Aut}(F_r); F(F_r)) \simeq \bigoplus_{i+j=n} \mathrm{Ext}^i_{\mathbf{gr}\text{-}\mathbf{Mod}}(\Lambda^j(\mathfrak{a}), F).$$

This is equivalent to the following statement, in term of homology, which is more suitable to present our methods.

Theorem 2. *Let $F : \mathbf{gr}^{op} \to \mathbf{Vec}_\mathbb{Q}$ be a polynomial functor of degree d and n, r non-negative integers such that $r \geq 2(n + d) + 3$. Then there is a natural isomorphism*

$$H_n(\mathrm{Aut}(F_r); F(F_r)) \simeq \bigoplus_{i+j=n} \mathrm{Tor}^{\mathbf{gr}}_i(F, \Lambda^j(\mathfrak{a})).$$

The left hand side in these theorems is called respectively n-th stable cohomology and n-th stable homology of automorphism groups of free groups with coefficients in F and will be denoted by $H^n_{st}(\mathrm{Aut}(F_\infty); F)$ and $H^{st}_n(\mathrm{Aut}(F_\infty); F)$. A better definition for $H^{st}_n(\mathrm{Aut}(F_\infty); F)$ (which holds also for non-polynomial

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functors) is $\operatorname{colim}_{r\in\mathbb{N}} H_n(\mathrm{Aut}(F_r); F(F_r))$; in fact, we will sketch the proof of theorem 2 with this definition of stable homology. The fact that it agrees with homology of $\mathrm{Aut}(F_r)$ for $r \geq 2(n+d)+3$ is an independent stability result due to Randal-Williams and Wahl [12].

Let us remind that a deep theorem of S. Galatius [7] implies that the stable homology with untwisted coefficients $H_*^{st}(\mathrm{Aut}(F_\infty); \mathbb{Q})$ is trivial. The previous theorem reduces to Galatius theorem for the constant functor $\mathbb{Q}$, but in fact it *uses* this theorem.

As a main computational consequence of theorem 2, we get, by using results of C. Vespa [16] (see also results by N. Kawazumi [8, 9] for the multiplicative structure):

Theorem 3. *For $n \geq 2(i+d)+3$, the cohomology group*

$$H^i(\mathrm{Aut}(F_n); \mathrm{Sym}^d((F_n)_{ab}) \otimes \mathbb{Q})$$

is 0, except when $i = d = 0$ or $i = d = 1$, where it is isomorphic to $\mathbb{Q}$.

For $n \geq 2(i+d)+3$, the dimension of the $\mathbb{Q}$-vector space

$$H^i(\mathrm{Aut}(F_n); \Lambda^d((F_n)_{ab}) \otimes \mathbb{Q})$$

equals the number of partitions of d when $i = d$ and 0 else.

Moreover, the bigraded $\mathbb{Q}$-algebra of stable cohomology of automorphism groups of free groups with coefficients in the exterior algebra over the rationalised abelianisation is a symmetric algebra on generators h_n, for n a positive integer, in bidegree (n, n).

This was conjectured by Randal-Williams ([11], corollary 6.4, proved under several conjectures involving topological methods different from the algebraic methods used here).

In low degree (≤ 2), some special cases were already known, due to Satoh and Kawazumi.

2 Covariant, contravariant and bivariant coefficients for $GL_n(\mathbb{Z})$ and $\mathrm{Aut}(F_n)$

Here we need a more general notion of stable homology of automorphism groups of free groups with twisted coefficients.

Let $\mathcal{G}$ be the following category: its objects are the free groups $\mathbb{Z}^{*n} = F_n$ (n being a non-negative integer) and the morphisms $G \to H$ are pairs (u, T) where $u : G \to H$ is a group monomorphism and T a subgroup of H such that $H = u(G) * T$ ($*$ denoting the free product), the composition being the obvious one. This category, introduced in [5] (following the setting of [4]), is a *homogeneous category* in the sense of Randal-Williams and Wahl [12]. We have canonical morphisms of $\mathcal{G}$ of the form $\mathbb{Z}^{*n} \to \mathbb{Z}^{*n} * \mathbb{Z} = \mathbb{Z}^{*(n+1)}$ which allow to define

$$H_*^{st}(\mathrm{Aut}(F_\infty); F) := \operatorname{colim}_{n\in\mathbb{N}} H_*(\mathrm{Aut}(\mathbb{Z}^{*n}); F(\mathbb{Z}^{*n}))$$

for each functor F from $\mathcal{G}$ to **Ab** (abelian groups).

We have an obvious functor $\mathcal{G} \to \mathbf{gr}$, which is identity on objects and maps an arrow (u, T) on u, but also a canonical functor $\mathcal{G}^{op} \to \mathbf{gr}$ which is also identity on objects and maps $(u, T) : G \to H$ to the morphism $H = u(G) * T \twoheadrightarrow u(G) \xrightarrow{\simeq} G$.

Most of the interesting coefficient systems for the groups $\mathrm{Aut}(\mathbb{Z}^{*n})$ factor in fact through the group epimorphism $\mathrm{Aut}(\mathbb{Z}^{*n}) \twoheadrightarrow GL_n(\mathbb{Z})$ induced by abelianisation. So, it makes very natural to compare our problem with the one of stable homology of general linear groups with polynomial coefficients, which was related to functor homology (for each ground ring) by the work of Scorichenko ([13]; for a published presentation of his results, see [2] § 5.2).

Let us introduce the usual category $\mathbf{ab}$ of finitely generated free abelian groups (which is a full subcategory of $\mathbf{Ab}$) and the (homogeneous) category $\mathcal{G}_{ab}$ with the same objects but with morphisms $U \to V$ pairs (f, C) where $f : U \to V$ is a monomorphism of abelian groups and C a subgroup of V such that $V = f(U) \oplus C$. As before we have canonical functors $\mathcal{G}_{ab} \to \mathbf{ab}$ and $\mathcal{G}_{ab}^{op} \to \mathbf{ab}$ which are identity on objects and we can define

$$H_*^{st}(GL_\infty(\mathbb{Z}); F) := \operatorname*{colim}_{n \in \mathbb{N}} H_*(GL_n(\mathbb{Z}); F(\mathbb{Z}^n))$$

for each functor $F : \mathcal{G}_{ab} \to \mathbf{Ab}$.

Scorichenko theorem relates $H_*^{st}(GL_\infty(\mathbb{Z}); F)$, when $F : \mathbf{ab}^{op} \times \mathbf{ab} \to \mathbf{Ab}$ is a polynomial (bi)functor [1], to $H_*^{st}(GL_\infty(\mathbb{Z}); \mathbb{Z})$ and homological algebra over the category $\mathbf{ab}$ (to be more precise, to Hochschild homology of this category with coefficients in F). A corollary of this result (which was proved before by Betley [1]) is that $H_*^{st}(GL_\infty(\mathbb{Z}); F) = 0$ when F is a polynomial functor $\mathbf{ab} \to \mathbf{Ab}$, or $\mathbf{ab}^{op} \to \mathbf{Ab}$, which is reduced (that is, which maps 0 to 0).

For automorphism groups of free groups, [5] proves that $H_*^{st}(\mathrm{Aut}(F_\infty); F) = 0$ when $F : \mathbf{gr} \to \mathbf{Ab}$ (*covariant coefficients*) is a reduced polynomial functor (see Christine Vespa's talk). But it does not hold for reduced polynomial functors $\mathbf{gr}^{op} \to \mathbf{Ab}$ (*contravariant coefficients*). This difference with the abelian situation illustrates that we can not find group automorphisms of $\mathrm{Aut}(F_n)$ making the following diagram to commute.

$$\begin{array}{ccc} \mathrm{Aut}(F_n) & \dashrightarrow & \mathrm{Aut}(F_n) \\ \downarrow & & \downarrow \\ GL_n(\mathbb{Z}) & \xrightarrow{g \mapsto {}^t g^{-1}} & GL_n(\mathbb{Z}) \end{array}$$

For polynomial *bi*functors on $\mathbf{gr}$ (*bivariant coefficients*), we have still partial results (in progress — see announcement in [3]), but the situation for contravariant polynomial functors — that is, $H_*^{st}(\mathrm{Aut}(F_\infty); F)$ for $F : \mathbf{gr}^{op} \to \mathbf{Ab}$ polynomial — is already rich and quite harder than Betley's cancellation in the abelian setting.

3 Strategy of proof

A first step is quite formal from the fact that $\mathcal{G}$ is a homogeneous category (see the formalism of the beginning of [4]): for any functor F on $\mathcal{G}$, we have a natural

[1] Here, we omit to indicate the precomposition by the canonical functor $\mathcal{G}_{ab} \to \mathbf{ab}^{op} \times \mathbf{ab}$ before F (to be consistent with the previous notation); we allow similar omissions in the sequel.

isormorphism

$$H_*^{st}(\mathrm{Aut}(F_\infty);F) \simeq H_*(\mathcal{G}\times \mathrm{Aut}(F_\infty);F)$$

where $\mathrm{Aut}(F_\infty)$ acts *trivially* on the right hand side; in particular, if F takes values in $\mathbb{Q}$-vector spaces, Galatius' cancellation of $\tilde{H}_*(\mathrm{Aut}(F_\infty);\mathbb{Q})$ implies that $H_*^{st}(\mathrm{Aut}(F_\infty);F) \simeq H_*(\mathcal{G};F)$.

After, we play with several categories of free groups. Let us remind before some general facts about Kan extensions. For any functor $\phi : \mathcal{C} \to \mathcal{D}$ between small categories, the precomposition functor $\phi^* = - \circ \phi : \mathbf{Mod}\text{-}\mathcal{D} \to \mathbf{Mod}\text{-}\mathcal{C}$, where $\mathbf{Mod}\text{-}\mathcal{C}$ (resp. $\mathcal{C}\text{-}\mathbf{Mod}$) is the category of functors from $\mathcal{C}^{op}$ (resp. $\mathcal{C}$) to $\Bbbk$-modules, $\Bbbk$ being a ground commutative ring ($\mathbb{Z}$ or $\mathbb{Q}$, for us), has a left adjoint $\phi_! : \mathbf{Mod}\text{-}\mathcal{C} \to \mathbf{Mod}\text{-}\mathcal{D}$ which satisfies a canonical isomorphism

$$\phi_!(F) \underset{\mathcal{D}}{\otimes} G \simeq F \underset{\mathcal{C}}{\otimes} \phi^* G;$$

we can derive it to get a Grothendieck spectral sequence

$$E^2_{i,j} = \mathrm{Tor}_i^{\mathcal{D}}(\mathbf{L}_j(\phi_!)(F), G) \Rightarrow \mathrm{Tor}^{\mathcal{C}}_{i+j}(F, \phi^* G).$$

In particular, for F constant, we get a spectral sequence

$$E^2_{i,j} = \mathrm{Tor}_i^{\mathcal{D}}(\mathbf{L}_j(\phi_!)(\Bbbk), G) \Rightarrow H_{i+j}(\mathcal{C}; \phi^* G).$$

We need still a general notation. Let $\mathcal{C}$ be a category, we denote by $\mathbf{S}(\mathcal{C})$ the category with the same objects and

$$\mathbf{S}(\mathcal{C})(a,b) := \{\, (u,v) \in \mathcal{C}(a,b)\times\mathcal{C}(b,a) \mid v\circ u = \mathrm{Id}_a\}.$$

If $\mathcal{C}$ is endowed with a symmetric monoidal structure $*$ whose unit 0 is a null object (so that we get canonical maps $a \to a*t$ and $a*t \to a$ in $\mathcal{C}$), we denote by $\mathbf{S}_c(\mathcal{C})$ the subcategory of $\mathbf{S}(\mathcal{C})$ with the same objects and as morphisms $a \to b$ the pairs (u,v) such that there exists a commutative diagram

$$\begin{array}{ccccc} a & \xrightarrow{u} & b & \xrightarrow{v} & a \\ & \searrow & \downarrow \simeq & \nearrow & \\ & & a*t & & \end{array}$$

The proof of theorem 2 consists now of three steps, about which a few details are given in the last three parts of this talk:

1. a homological comparison between $\mathbf{S}_c(\mathbf{gr})$ and $\mathbf{gr}^{op}$ with polynomial coefficients. To be more precise, let $\beta : \mathbf{S}_c(\mathbf{gr}) \to \mathbf{gr}^{op}$ be the canonical functor. We have:

   Theorem 4. *If F is a polynomial functor in* $\mathbf{Mod}$-$\mathbf{gr}$ *and G any functor in* $\mathbf{gr}$-$\mathbf{Mod}$*, then the natural morphism of graded abelian groups*

   $$\mathrm{Tor}_*^{\mathbf{S}_c(\mathbf{gr})}(\beta^* G, \beta^* F) \to \mathrm{Tor}_*^{\mathbf{gr}}(F,G)$$

   induced by β is an isomorphism.

2. Show that the derived Kan extension of the canonical functor $\gamma : \mathcal{G} \to \mathbf{S}_c(\mathbf{gr})$ is given by

$$\mathbf{L}_\bullet(\gamma_!)(\Bbbk)(A) \simeq H_\bullet(\Omega^\infty \Sigma^\infty(B(A)); \Bbbk)$$

($B(A)$ denoting the classifying space of the group A); in particular, in characteristic zero:

$$\mathbf{L}_\bullet(\gamma_!)(\mathbb{Q})(A) \simeq \Lambda^\bullet(A_{ab} \otimes \mathbb{Q});$$

3. From the previous results, we get a natural spectral sequence

$$E^2_{i,j} = \mathrm{Tor}_i^{\mathbf{gr}}(F, \Lambda^j(\mathfrak{a})) \Rightarrow H^{st}_{i+j}(\mathrm{Aut}(F_\infty); F)$$

(remind that $\mathfrak{a}$ denotes the rationalised abelianisation functor) for $F : \mathbf{gr}^{op} \to \mathbf{Vec}_\mathbb{Q}$ polynomial. One gets the conclusion by showing that the spectral sequence collapses at the second page and that the grading associated to its E^∞-page is trivial, what is made by a formality argument.

Before sketching the proofs of these steps, let us say a few words about the proof of Scorichenko theorem for stable homology of general linear groups over $\mathbb{Z}$. For the same formal reasons as above, one has a natural isomorphism $H^{st}_*(GL_\infty(\mathbb{Z}); F) \simeq H_*(\mathcal{G}_{ab} \times GL_\infty(\mathbb{Z}); F)$ (with trivial action of the linear group on the right hand side) for any functor F in $\mathcal{G}_{ab}$-**Mod**. The homological comparison between $\mathbf{S}_c(\mathbf{ab})$ and $\mathbf{ab}^{op}$ works in the same way as the first step above. But here the canonical functor $\mathcal{G}_{ab} \to \mathbf{S}_c(\mathbf{ab}) (= \mathbf{S}(\mathbf{ab}))$ is an *equivalence*: we are done only with the comparison between $\mathbf{S}_c(\mathbf{ab})$ and $\mathbf{ab}^{op}$. The second step above has no analogue in this abelian context; in some sense, it is a way to control how the free product (=categorical sum) of groups differs from being a categorical product.

4 Scorichenko's machine

The idea to prove theorem 4 is a very general, simple and powerful one, due to Scorichenko [13].

Let $\mathcal{C}$ be a (small) category with finite coproducts (denoted by $*$) and a zero object. If E is a finite set and I a subset of E, let us denote by t_E the endofunctor of $\mathcal{C}$-**Mod** given by precomposition by $a \mapsto a^{*E}$ and by $u^I_E : t_E \to \mathrm{Id}$ the natural transformation given by precomposition by the natural morphism $a^{*E} \to a$ whose component $a \to a$ corresponding to the factor labelled by $e \in E$ is identity if $e \in I$ and 0 else. If (E, e) is a finite pointed set, one defines the *cross-effect*[2] $cr_{(E,e)}$ as the natural transformation

$$cr_{(E,e)} := \sum_{e \in I \subset E} (-1)^{\mathrm{Card}(I)-1} u^I_E : t_E \to \mathrm{Id}.$$

A functor F of $\mathcal{C}$-**Mod** is *polynomial of degree* $\leq d$ if and only if $cr_{(E,e)}(F) = 0$, where E is a set of cardinality $d + 2$.

[2] *Warning:* usually, cross-effects are defined as endofunctors of functor categories (and not natural transformations). The fact that the definition of polynomial functors with cross-effects as natural transformations is equivalent to the usual one (with usual cross-effects) is an exercise, but this change of point of view is a key point of Scorichenko's work.

Let us denote by $\mathbf{E}(\mathbf{gr})$ the subcategory of $\mathbf{gr}$ with the same objects and whose morphisms are epimorphisms isomorphic to an epimorphism $S * T \twoheadrightarrow S$. If F and G are functors defined on $\mathbf{gr}$, we say that a collection of morphisms $F(a) \to G(a)$ defined for $a \in \mathrm{Ob}\,\mathbf{gr}$ is *weakly natural* if it is natural with respect to $\mathbf{E}(\mathbf{gr})$.

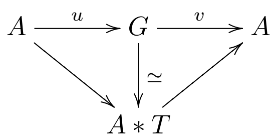
Proposition 5. *Let $T \in \mathbf{gr}$-$\mathbf{Mod}$, $d \in \mathbb{N}$ and (E, e) a pointed set of cardinality $d+2$. Assume that the natural transformation $cr_{(E,e)}(T) : t_E(T) \to T$ is an epimorphism having a weakly natural splitting. Then $\mathrm{Tor}^{\mathbf{gr}}_*(F, T) = 0$ for all functor F in $\mathbf{Mod}$-$\mathbf{gr}$ polynomial of degree $\leq d$.*

Scorichenko's original criterium is the same statement with $\mathbf{gr}$ replaced by a (small) additive category. The proof is almost the same. Note that this does *not* work with $\mathbf{gr}$ replaced by any small category with finite coproducts and a zero object: we need in the proof the fact that the natural transformations u^I_E defined above are already defined on $\mathbf{E}(\mathbf{gr})$, what is not completely formal (it uses "triangular automorphisms" of free groups).

5 The classifying space of the category $\mathrm{C}(A)$

For formal reasons, $\mathbf{L}_\bullet(\gamma_!)(\Bbbk)(A)$ identifies with $H_\bullet(\mathrm{C}(A); \Bbbk)$ where $\mathrm{C}(A)$ is the comma category associated to the functor $\gamma^*\mathbf{S}_c(\mathbf{gr})(A, -) : \mathcal{G} \to \mathbf{Sets}$.

Concretely, objects of $\mathrm{C}(A)$ are triples $(G, A \xrightarrow{u} G, G \xrightarrow{v} A)$ where G is an object of $\mathcal{G}$ (that is, a free group of finite rank), u and v are group morphisms such that $v \circ u = \mathrm{Id}_A$ and there exists a commutative diagram



(but this diagram is *not* given in the structure); morphisms in $\mathrm{C}(A)$ are morphisms in $\mathcal{G}$ satisfying the obvious compatibility conditions.

Let $(G, A \xrightarrow{u} G, G \xrightarrow{v} A)$ be an object of $\mathrm{C}(A)$, consider the subgroup $K := \mathrm{Ker}(v)$ of G. It comes with an action of A (conjugation through the group monomorphism u). Moreover, K is a *finitely generated free A-group*: there exists a finitely generated free group H such that

$$K \simeq \mathop{\bigstar}_{a \in A} {}^{a}H$$

(with the obvious A-action; in general, in the sequel, we denote with an exponent actions of the group A on other groups) as an A-group (but there is no canonical choice for H).

We introduce the following category, for an A-group K.

Definition 6. Let $\mathfrak{D}_A(K)$ be the category whose objects are finite (possibly empty) sequences $(T_1, \dots, T_r)$ of non-trivial subgroups of K such that the group K is the (internal) free product of the subgroups ${}^{a}T_i$ for $a \in A$ and $1 \leq i \leq r$ and whose morphisms $(T_1, \dots, T_r) \to (U_1, \dots, U_s)$ consist of a set-epimorphism $\varphi : \mathbf{s} := \{1, \dots, s\} \to \mathbf{r}$ and elements $a_1, \dots, a_s$ of A such that

$$T_i = \mathop{\bigstar}_{\varphi(j)=i} {}^{a_j}U_j$$

for all index i.

(Here and below in similar definitions, we do not precise how to compose morphisms: the reader can guess it or refer to [3].)

It is easy to see that this category is a partially preordered set (that is, there is at most one morphism between two objects).

Theorem 7. *If K is a finitely generated free A-group, then the category $\mathfrak{D}_A(K)$ is contractible.*

Surprisingly, the proof consists of two independent parts, one to show that this category is connected (it relies on a variation for A-groups of Nielsen transformations) and the other to show that its homotopy type is discrete (it is a rather general argument, relying on the "rigidity" of free products).

If $\underline{G} = (G, A \xrightarrow{u} G, G \xrightarrow{v} A)$ is an object of $\mathrm{C}(A)$ and K is the A-group defined above, let $\mathcal{D}_A(\underline{G})$ denote the category $\mathfrak{D}_A(K)$.

One sees easily that $\mathcal{D}_A$ defines an oplax functor from $\mathrm{C}(A)$ to small categories (in fact, to partially preordered sets). It allows to form the *Grothendieck construction* $\mathrm{C}(A) \int \mathcal{D}_A$. The contractibility of the values of $\mathcal{D}_A$ implies that the canonical functor $\mathrm{C}(A) \int \mathcal{D}_A \to \mathrm{C}(A)$ is a homotopy equivalence (see [15]).

It is quite simpler to determine the homotopy type of $\mathrm{C}(A) \int \mathcal{D}_A$ than the one of $\mathrm{C}(A)$, because the category $\mathrm{C}(A) \int \mathcal{D}_A$ is equivalent (it is an easy game between internal and external free products of groups) to another Grothendieck construction, that one can concretely describe as follows: its objects are finite sequences $(G_1, \dots, G_n)$ of non-trivial free groups of finite rank, and morphisms $(G_1, \dots, G_n) \to (H_1, \dots, H_m)$ consist of a *partially defined* surjection $\varphi : \mathbf{m} \to \mathbf{n}$, group isomorphisms $G_i \simeq \bigstar_{\varphi(j)=i} H_j$ for each index i and elements $a_1, \dots, a_m$ of A.

We have now to show that this category has the same homotopy type (with the right functoriality in A) as the infinite loop space $\Omega^\infty \Sigma^\infty B(A)$. This is made by using general properties of Segal's machinery [14] associating a spectrum to each symmetric monoidal category, and Galatius theorem [7] showing that the canonical functor from the symmetric monoidal groupoid of finite sets to the symmetric monoidal groupoid of finitely generated free groups induces a homotopy equivalence between the corresponding spectra. (The intervention of Galatius theorem in this part of the proof was quite unexpected for the author, contrary to its use at the beginning of the proof, when $\mathcal{G} \times \mathrm{Aut}(F_\infty)$ arises.)

6 Formality argument

This final part of the proof is inspired by Pirashvili's proof of Hodge decomposition for Hochschild homology by functor homology (see [10]), which relies on Dold's obstruction theory for chain complexes [6]. All comes from the cancellation of

$$\mathrm{Ext}^{m-n+1}_{\mathbf{gr}\text{-}\mathbf{Mod}}(\Lambda^n(\mathfrak{a}), \Lambda^m(\mathfrak{a})),$$

which implies the same cancellation (by an obvious variant of theorem 4) for Ext-groups on $\mathbf{Mod}\text{-}\mathbf{S}_c(\mathbf{gr})$ (using precomposition by $\gamma : \mathbf{S}_c(\mathbf{gr}) \to \mathbf{gr}^{op}$). This cancellation is included in Vespa's paper [16] (in fact, the same cancellation holds, even on $\mathbb{Z}$, with *tensor*, instead of *exterior*, powers, and here we work on $\mathbb{Q}$).

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