Homological stability for automorphism groups of free products *via* polynomial functors

Notes of a talk given for Copenhagen Masterclass on Homological stability, August 26-30, 2013

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Warning: these notes may not faithfully coincide with the corresponding talk. They may also contain misprints and I am sorry for this; if you remark anything bad in this text I will be happy to correct it (please send an email to aurelien.djament@univ-nantes.fr).
The references of these notes are highly incomplete, please see the references of [2].

The aim of this talk is to give an idea of the proof of the following theorem, which is the main result of [2].

Theorem 1 (Collinet-Djament-Griffin). *For any group G which is indecomposable for the free product $*$ and not isomorphic to $\mathbb{Z}$ (for example, a finite group) and any non-negative integers n, i such that $n \geq 2i + 2$, the canonical map*

$$H_i\big(\mathrm{Aut}(G^{*n});\mathbb{Z}\big) \to H_i(\mathrm{Aut}\big(G^{*n+1});\mathbb{Z}\big)$$

is an isomorphism.

It answers positively, in many cases, a conjecture of A. Hatcher and N. Wahl (they conjectured that the result holds for all groups G). They proved this for $G = \mathbb{Z}/n$, $n \in \{2, 3, 4, 6\}$ in [4] by using geometric and topological methods. Even if Theorem 1 deals with untwisted coefficients, our main tool to prove this is the notion of polynomial functor. This result is a special case of a more general theorem dealing with some subgroups of automorphism groups of arbitrary finite free products.

Contents

1 Recollections on homological stability and functor homology

Let $(\mathcal{C}, *, 0)$ be a small symmetric monoidal category whose unit 0 is an initial object and A an object of $\mathcal{C}$. So we have, for any $n \in \mathbb{N}$ and any functor $F : \mathcal{C} \to \mathbf{Ab}$, a group morphism $\mathrm{Aut}_{\mathcal{C}}(A^{*n}) \to \mathrm{Aut}_{\mathcal{C}}(A^{*n+1})$ and an equivariant map $F(A^{*n}) \to F(A^{*n+1})$ (both induced by the inclusion of the n first factors of A^{*n} in A^{*n+1}) and, for any $i \in \mathbb{N}$, an induced sequence of homology groups

$$\dots \to H_i\big(\mathrm{Aut}_{\mathcal{C}}(A^{*n}); F(A^{*n})\big) \to H_i\big(\mathrm{Aut}_{\mathcal{C}}(A^{*n+1}); F(A^{*n+1})\big) \to \dots$$

(see Christine Vespa's lectures); we can wonder whether these maps are isomorphism for n big enough (depending on i). Here we have no more functoriality in n, because the inner automorphisms of a group act trivially in homology. To use methods of polynomial functors and functor homology to study this kind of problem could help a little, at least to make clear why it is often almost free to get homological stability for a polynomial functor F when we can prove it for the constant functor $F = \mathbb{Z}$ (the first approach of homological stability with twisted polynomial coefficients goes back to Dwyer [3]). But the key ingredient for homological stability proofs is the high acyclicity of appropriate complexes with action of our groups, with stabilizers given by smaller groups (or related groups) of the family whose homological stability we want to prove.

Nevertheless, the use of functor categories can help in a stronger way for homological stability questions (see Tom Church's lectures) when dealing with *congruence groups*. Suppose that $(\mathcal{D}, +, 0)$ is another small symmetric monoidal category and that $\Phi : \mathcal{C} \to \mathcal{D}$ is a strict monoidal functor. For any object X of $\mathcal{C}$, we can define the congruence group

$$\Gamma_\Phi(X) := \mathrm{Ker}\,\big(\mathrm{Aut}_{\mathcal{C}}(X) \xrightarrow{\Phi_*} \mathrm{Aut}_{\mathcal{D}}(\Phi(X))\big).$$

If we look at the homology groups $H_*(\Gamma_\Phi(X))$ (we do not indicate always the coefficients when they are equal to $\mathbb{Z}$), they inherit an action of the image of Φ_*; so $X \mapsto H_*(\Gamma_\Phi(X))$ gives a generally highly non trivial functor from some category (that we do not precise [1]; the only important feature is that it can not be reduced to a sequence of abelian groups as before: we have a richer structure) obtained from $\mathcal{C}$ and Φ.

Usually (for example, for congruence groups in the usual sense), the congruence groups defined above are much harder to study (for homology and other questions) that the automorphism groups in which they live. But for the situation of automorphism groups of free products as in the statement of Theorem 1 (we will make precise further the good functoriality setting), we get groups which are easier, so that a natural approach to study homological stability for automorphism groups of free products is to study first the functorial properties of the homology of these "congruence groups" — which are known as *Fouxe-Rabinovitch groups* — and going back to our groups through the Hochschild-Serre spectral sequence giving their homology from the homology of these groups (the quotient being easily understood from symmetric groups, whose homology — including stability — has been studied for a long time). It is the very general idea of the proof of Theorem 1.

[1] Under standard assumptions, this category is $\mathcal{D}$.

2 Recollections on automorphism groups of free products

As we can define elementary matrices in general linear groups (or more generally, elementary automorphisms for direct sums of objects in an additive category), we have some "obvious" group automorphisms for any free product $\mathrm{G} = \underset{i \in I}{\star} G_i$ of groups:

1. *permutation of isomorphic factors*: for simplicity, suppose that we have choosen an isomorphism between G_i and G_j when these groups are isomorphic (so that we can identify them); let us note by S the set of equivalence classes of I for the relation $i \sim j$ when $G_i \simeq G_j$. Then we have a group morphism
$$\Sigma := \prod_{A \in S} \mathfrak{S}(A) \to \mathrm{Aut}(\mathrm{G})$$
(which is a monomorphism if the G_i are non-trivial);

2. *automorphisms of individual factors*: we have an obvious group monomorphism
$$\mathrm{aut}(G) := \prod_{i \in I} \mathrm{Aut}(G_i) \to \mathrm{Aut}(\mathrm{G});$$

3. *partial conjugations*: let $i \neq j$ be two elements of I and x an element of G_i. The group morphism
$$\alpha_{i,j}(x) : \mathrm{G} \to \mathrm{G} \qquad g \in G_t \mapsto \begin{cases} g & \text{if} \quad t \neq j \\ xgx^{-1} & \text{if} \quad t = j \end{cases}$$
is an automorphism; moreover we get in this way a group morphism
$$\alpha_{i,j} : G_i \to \mathrm{Aut}(\mathrm{G})$$
(which is a monomorphism if G_j is non-trivial).

From now, we suppose that all groups G_i are non-trivial.

The way in which the automorphisms of the two first kinds interact is clear: we get a group monomorphism
$$\Sigma \ltimes \mathrm{aut}(G) \hookrightarrow \mathrm{Aut}(\mathrm{G})$$
where Σ acts on the other group by permutation of factors corresponding to isomorphic groups. It is completely similar to the matrix situation, in which we get a group monomorphism from the semi-direct product of diagonal matrices by permutation matrices into the general linear group.

The subgroup of $\mathrm{Aut}(\mathrm{G})$ generated by the image of all group monomorphisms $\alpha_{i,j}$ is called *Fouxe-Rabinovitch subgroup* of $\mathrm{Aut}(\mathrm{G})$ and denoted by $FR(G)$ (where $G = (G_i)_{i \in I}$), it is much harder to understand. There are nevertheless some classical "obvious" relations between these morphisms $\alpha_{i,j}$ (analogous to the Steinberg relations for elementary matrices), and in fact we get a presentation of $FR(G)$ with these only relations. So, the situation is easier than in the linear one, where the Steinberg group differs from the group generated by elementary matrices by the abelian group K_2.

Let us call *symmetric automorphism* of G (relative to the decomposition $\mathrm{G} = \underset{i\in I}{\star} G_i$) an automorphism φ such that for any $i \in I$ there exists $j \in I$ such that $\varphi(G_i)$ is conjugated to G_j; j is then unique and $i \mapsto j$ is a permutation in Σ. The symmetric automorphisms form a subgroup $\Sigma\mathrm{Aut}(G)$ of Aut(G) which is easily seen to be isomorphic to

$$\Sigma\mathrm{Aut}(G) \simeq \Sigma \ltimes \mathrm{aut}(G) \ltimes FR(G) \tag{1}$$

(the map assigning $i \mapsto j$ to φ as above is a section of the inclusion $\Sigma \hookrightarrow \Sigma\mathrm{Aut}(G)$).

An easy and classical consequence of Kurosh theorem is the following:

Theorem 2. *Suppose that the groups G_i are indecomposable for the free product and not isomorphic to $\mathbb{Z}$. Then the inclusion*

$$\Sigma\mathrm{Aut}(G) \subset \mathrm{Aut(G)}$$

is an equality.

(When there are factors isomorphic to $\mathbb{Z}$, the situation is more flexible: beside the partial conjugations, we have left and right multiplication on a fixed factor by a generator of another factor. Homological stability for automorphism groups of free groups is a deep result — quite harder than the result about that I am talking — which was established during the nineties by Hatcher, Vogtmann and Wahl.)

Theorem 1 is a consequence of Theorem 2 and an homological stability theorem for symmetric automorphisms of arbitrary finite free products.

3 Functoriality of Fouxe-Rabinovitch groups and reduction to a polynomial property

Let Γ be the category whose objects are finite sets and morphism *partial* functions, that is: a morphism $A \to B$ in Γ is a pair (E, f) where E is a subset of A and $f : E \to B$ a function. (This category is equivalent to the category of finite pointed sets, the morphism being as usual the functions which send the base-point of the source on the base-point of the target; an equivalence from this category of pointed sets to Γ is obtained on objects by removing the base-point and on morphisms by assigning to a pointed function $f : X \to Y$ the partial function defined on the complement of the preimage of the base-point of Y that it induces.) If $\mathcal{C}$ is any category, we define a new category $\Gamma \int \mathcal{C}$ as follows:

1. the objects of $\Gamma \int \mathcal{C}$ are pairs $(A, (C_a)_{a\in A})$ where A is an object of Γ and (C_a) a family of objects of $\mathcal{C}$ labelled by A;
2. the morphisms $(A, (C_a)_{a\in A}) \to (B, (D_b)_{b\in B})$ in $\Gamma \int \mathcal{C}$ are pairs

   $$\big(u = (E, f : E \to B), (\gamma_a : C_a \to D_{f(a)})_{a\in E}\big)$$

   where u is a morphism of Γ and the γ_a are morphisms of $\mathcal{C}$;
3. the composition is obtained from the compositions of Γ and $\mathcal{C}$ in an obvious way.

If Γ' is a subcategory of Γ, we denote by $\Gamma' \int \mathcal{C}$ the subcategory of $\Gamma \int \mathcal{C}$ whose objects and morphisms are those whose image by the obvious projection functor $\Gamma \int \mathcal{C} \to \Gamma$ lies in Γ'.

We are particularly interested by the following subcategories [2] of Γ, having the same objects as Γ: the subcategory $\widetilde{\Theta}$ of partial *injections*, the subcategory Θ of injections (everywhere defined) and the category Σ of bijections (everywhere defined).

Observation: if $(\mathcal{C}, *, 0)$ is a symmetric monoidal category whose unit 0 is a zero object [3] (example: groups with the free product), we have a canonical functor

$$\widetilde{\Theta} \int \mathcal{C} \to \mathcal{C} \qquad (A, (C_a)_{a \in A}) \mapsto \mathop{\star}_{a \in A} C_a$$

which is defined on morphisms by inclusions of "factors" (for $*$) and deleting factors which do not lie in the defining set of the underlying morphism of $\widetilde{\Theta}$ (to include factors, we need 0 to be an initial object — the prototype of these morphisms is $C \simeq C * 0 \to C * D$ — and to delete factors we need 0 to be a terminal object — the prototype of these morphisms is $C * D \to C * 0 \simeq C$).

To assign to a group G its automorphism group does not define a functor on the category **Gr** of groups (the only clear functoriality is on the subcategory of groups with group isomorphisms). But for any $n \in \mathbb{N}$, we get a functor

$$FR_n : \mathbf{Gr}^n \to \mathbf{Gr} \qquad (G_1, \dots, G_n) \mapsto FR(G_1, \dots, G_n).$$

Moreover, these functors assemble when n changes to give a functor

$$FR : \widetilde{\Theta} \int \mathbf{Gr} \to \mathbf{Gr}.$$

The effect of this functor on a generating element $\alpha_{i,j}(x)$ of $FR(G)$ is given as follows: if $\big(u = (E, f : E \to B), (\gamma_a : C_a \to D_{f(a)})_{a \in E}\big)$ is a morphism of $\widetilde{\Theta} \int \mathbf{Gr}$, this element is killed if i or j does not lie in E, and is sent on $\alpha_{f(i), f(j)}(\gamma_i(x))$ else. To prove that this construction is well-defined is easy with the standard presentation of $FR(G)$.

Let us look back to the isomorphism (1): $\Sigma \ltimes \mathrm{aut}(G)$ is exactly the automorphism group of G in the category $\widetilde{\Theta} \int \mathbf{Gr}$, and its action on $FR(G)$ that is used to build the semi-direct product is just the action by functoriality.

Cross effect and polynomial functors

Let Σ be the subcategory of isomorphisms in Γ. For any category $\mathcal{C}$, there is an exact functor

$$cr : \mathbf{Fct}(\widetilde{\Theta} \int \mathcal{C}, \mathbf{Ab}) \to \mathbf{Fct}(\Sigma \int \mathcal{C}, \mathbf{Ab})$$

called *cross effect* (this notion goes back to Eilenberg and Mac Lane in the fifties) such that there exists a functorial isomorphism

$$F(C) \simeq \bigoplus_{A \subset E} cr(F)(C|_A)$$

[2] *Warning:* these notations are not exactly the same as in the article [2].

[3] If 0 is only an initial object, we get similarly a functor $\Theta \int \mathcal{C} \to \mathcal{C}$.

where $F : \widetilde{\Theta} \int \mathcal{C} \to \mathbf{Ab}$ is a functor and $C = (C_e)_{e\in E}$ an object of $\widetilde{\Theta} \int \mathcal{C}$; we have denoted by $C|_A$ the object $(C_e)_{e\in A}$ of $\Sigma \int \mathcal{C}$. (There are several ways to define the functor cr, for example by using nice commuting idempotents coming from the elementary combinatorics of subsets of a given finite set, but it is not very important here.)

For any integer d, we say that the functor $F : \widetilde{\Theta} \int \mathcal{C} \to \mathbf{Ab}$ is *polynomial with degree at most d* if $cr(F)(C) = 0$ when $C = (C_e)_{e\in E}$ with $\mathrm{Card}(E) > d$.

In the last part of this talk, we will give a few words about the proof of the following theorem:

Theorem 3. *For any $i \in \mathbb{N}$, the functor*

$$\widetilde{\Theta} \int \mathbf{Gr} \xrightarrow{FR} \mathbf{Gr} \xrightarrow{H_i} \mathbf{Ab}$$

is polynomial with degree at most $2i$.

Now let us explain how to deduce Theorem 1 from this result.

Homological stability for symmetric groups with twisted coefficients

The homology of symmetric groups was computed by Nakaoka in the early sixties; in particular we have the following stability result (that one can get also without computing this homology):

Theorem 4. *For any abelian group M and any integers i and n, the canonical map*

$$H_i(\mathfrak{S}_n; M) \to H_i(\mathfrak{S}_{n+1}; M)$$

(where the symmetric groups act trivially on M) is an isomorphism if $n \geq 2i+1$.

As observed by Betley (see [1], section 4), it implies easily a twisted homology stability result:

Corollary 1. *If $F : \widetilde{\Theta} \to \mathbf{Ab}$ is a polynomial functor of degree d, the canonical map*

$$H_i(\mathfrak{S}_n; F(\mathbf{n})) \to H_i(\mathfrak{S}_{n+1}; F(\mathbf{n+1}))$$

is an isomorphism for $n > 2i + d$.

(Here $\mathbf{n} := \{1, \dots, n\}$.)

Proof. The natural decomposition

$$F(E) \simeq \bigoplus_{A\subset E} cr(F)(A)$$

and the fact that $cr(F)(A) = 0$ for $\mathrm{Card}(A) > d$ give an isomorphism

$$F(\mathbf{n}) \simeq \bigoplus_{l\leq d} \mathrm{Ind}_{\mathfrak{S}_l\times\mathfrak{S}_{n-l}}^{\mathfrak{S}_n}\big(cr(F)(\mathbf{l})\big)$$

of representations of $\mathfrak{S}_n$, where $\mathfrak{S}_l$ acts by functoriality on $cr(F)(\mathbf{l})$ and $\mathfrak{S}_{n-l}$ trivially. So, using Shapiro lemma and Künneth formula, we get an isomorphism

$$H_i(\mathfrak{S}_n; F(\mathbf{n})) \simeq \bigoplus_{l\leq d} H_i\big(\mathfrak{S}_l\times\mathfrak{S}_{n-l}; cr(F)(\mathbf{l})\big) \simeq \bigoplus_{\substack{l\leq d\\ r+s=i}} H_r\big(\mathfrak{S}_{n-l}; H_s(\mathfrak{S}_l; cr(F)(\mathbf{l}))\big)$$

so that Theorem 4 implies the result. $\square$

As a consequence, admitting Theorem 3, we get:

Theorem 5. *For any group G, the canonical map*

$$H_i(\Sigma \mathrm{Aut}_n(G)) \to H_i(\Sigma \mathrm{Aut}_{n+1}(G))$$

is an isomorphism for $n \geq 2i+2$.

(We have noted here $\Sigma\mathrm{Aut}_n(G)$ for $\Sigma\mathrm{Aut}(G,\dots,G)$ with G repeated n times. In fact, we get in [2] a more general statement, with different groups allowed in $\Sigma\mathrm{Aut}(G_1,\dots,G_n)$, following exactly the same principle, but we restrict for this talk to a single group to avoid technicalities.)

Proof. It is divided into two steps.

1. We prove that for $q \in \mathbb{N}$ the functor

$$\widetilde{\Theta} \to \mathbf{Ab} \qquad E \mapsto H_q\big((\mathrm{Aut}(G))^E \ltimes FR_E(G)\big)$$

   is polynomial with degree $\leq 2q$.

   By using the Hochschild-Serre spectral sequence and the fact that polynomial functors of degree $\leq d$ form, for each d, a thick subcategory, we see that it is enough to prove that

$$E \mapsto H_i\big((\mathrm{Aut}(G))^E; H_j(FR_E(G))\big)$$

   is polynomial with degree $\leq 2q$ when $i+j=q$. But this property follows from Theorem 3, the decomposition

$$H_j(FR_E(G)) \simeq \bigoplus_{A \subset E} cr(H_j(FR))_A(G)$$

   being equivariant with respect to the natural action of $\mathrm{Aut}(G)^E$ (the action on the factor corresponding to A being given through the projection $\mathrm{Aut}(G)^E \twoheadrightarrow \mathrm{Aut}(G)^A$ which kills the factors out of A).

2. Now, apply Corollary 1 to get that the stabilization map

$$E^2_{p,q}(n) \to E^2_{p,q}(n+1),$$

   where

$$E^2_{p,q}(n) := H_p\big(\mathfrak{S}_n; H_q\big((\mathrm{Aut}(G))^n \ltimes FR_n(G)\big)\big),$$

   is an isomorphism for $n > 2(p+q)$. As this map is the restriction to second pages of the map of Hochschild-Serre spectral sequences induced by the stabilization group morphism, we can finish the proof by formal arguments about spectral sequences (see [2] for the details).

□

4 Overview of the proof of the required polynomial property of $H_*(FR)$

Let $G = (G_i)_{i\in E}$, where E is a finite set, be a family of groups. We have a generalized diagonal functor

$$D_G : (\Gamma^E)^{op} \to \mathbf{Gr} \qquad (A_i)_{i\in E} \mapsto \prod_{i\in E} G_i^{A_i}.$$

There exist a poset J_E of trees labelled by E and a functor $F_E : J_E \to (\Gamma^E)^{op}$ such that the group $FR(G)$ is isomorphic to $\operatorname*{colim}_{J_E} D_G \circ F_E$ and there exist a spectral sequence

$$E^2_{p,q} = H_p(J_E, \mathcal{H}_q(D_G \circ F_E; \mathbb{Z})) \Rightarrow H_{p+q}(FR(G); \mathbb{Z}); \tag{2}$$

moreover all these data are functorial in the object G of $\widetilde{\Theta} \int \mathbf{Gr}$. Here $\mathcal{H}_q(T;\mathbb{Z})$, for any functor T from a category $\mathcal{C}$ to $\mathbf{Gr}$, means the composition of T with the functor $H_q(-;\mathbb{Z}) : \mathbf{Gr} \to \mathbf{Ab}$.

We will give no details on the precise signification of J_E, F_E and the proof of this result. It comes from a (variation around a) theorem due to Chen-Glover-Jensen about the contractibility of some complex with a nice action of $FR(G)$, the diagonal subgroups $D_G(F_E(A))$ appearing as vertex stabilizers.

To deduce Theorem 3 from the spectral sequence (2), we use still the notion of cross effect, rephrasing in a functorial way some constructions of J. Griffin's Ph.D.: we prove (by using some basic tools of functor homology) that for any functor $T : (\Gamma^E)^{op} \to \mathbf{Ab}$, there is a natural decomposition

$$H_0(J_E; T \circ F_E) \simeq \bigoplus_{A\in\mathcal{F}_E} cr(T)(F_E(A))$$

where $\mathcal{F}_E$ is some explicit subset of J_E, whereas

$$H_n(J_E; T \circ F_E) = 0 \quad \text{for} \quad n > 0.$$

As $\mathcal{H}_q \circ D_G$ is described from the homology of the groups G_i by Künneth formula, we can conclude by an easy direct combinatorical argument.

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