Noetherianity and homological finiteness of polynomial functors (I)

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After recent works with Antoine Touzé, but also more classical works by Teimuraz Pirashvili and Lionel Schwartz especially.

Very partial references

- Vincent Franjou, Jean Lannes and Lionel Schwartz. *Autour de la cohomologie de Mac Lane des corps finis*. Invent. Math. **115** (1994). *(Classical Schwartz's lemma: § 10).*
- Stanisław Betley and Teimuraz Pirashvili. *Twisted (co)homological stability for monoids of endomorphisms*. Math. Ann. 295 (1993)
- Aurélien Djament and Antoine Touzé. *Finitude homologique des foncteurs sur une catégorie additive et applications*. Trans. AMS **376** (2023).
- Aurélien Djament and Antoine Touzé. *Sur la noethérianité locale des foncteurs polynomiaux*. To appear in Tunisian Journal of Mathematics.

1. Polynomial functors over an additive category: motivations, history
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Functor categories: definitions and notations

Let $\mathcal{C}$ by an (essentially) small category and K a commutative ring. We denote by $\mathcal{F}(\mathcal{C};K)$ the category of functors from $\mathcal{C}$ to K-modules. One can think of this as the category of representations over K of $\mathcal{C}$ (of which one can think as a monoid with several objects).

$\mathcal{F}(\mathcal{C};K)$ is a nice abelian category (see below).

If E is a set, we denote by $K[E]$ the free K-module built on E. We denote by $P^c_{\mathcal{C}}$ the functor $K[\mathcal{C}(c,-)]$ of $\mathcal{F}(\mathcal{C};K)$, for $c \in \mathrm{Ob}\,\mathcal{C}$. By a variation around the Yoneda lemma,

$$\mathrm{Hom}_{\mathcal{F}(\mathcal{C};K)}(P^c_{\mathcal{C}}, F) \simeq F(c)$$

naturally in F and in c. These functors are important because they *generate* the category $\mathcal{F}(\mathcal{C};K)$ and are projective; they are analogues of free modules of finite rank in a module category.

Functor categories: case of an additive source

We are interested with the case when the source category is *additive*, i.e. has finite direct sums, what induces a structure of abelian group on the sets of morphisms between two given objects, so that the composition is bilinear.

A fundamental particular case is the one of the source category $\mathbf{P}(R)$ of finitely generated projective left R-modules over a ring R (one can replace it by the subcategory of free modules without changing the functor category); one will note $\mathcal{F}(R, K)$ for $\mathcal{F}(\mathbf{P}(R); K)$.

If $\mathcal{A}$ is a small additive category, *additive* functors from $\mathcal{A}$ to K-**Mod** form a full abelian subcategory $\mathbf{Add}(\mathcal{A}; K)$ of $\mathcal{F}(\mathcal{A}; K)$, which is generally much easier to understand than $\mathcal{F}(\mathcal{A}; K)$.

Example: the functor mapping a bimodule V on the additive functor $V \underset{R}{\otimes} -$ defines an equivalence between the category of (K, R)-bimodules (i.e. left $K \underset{\mathbb{Z}}{\otimes} R^{\mathrm{op}}$-modules) and $\mathbf{Add}(\mathbf{P}(R); K)$.

Eilenberg and MacLane (1954) introduced the notion of *polynomial functor* from an additive category to a module category, that we will remind later. This is an extension of the notion of additive functor: polynomial functors of degree at most 0 are constant functors; polynomial functors of degree at most 1 are direct sums of a constant functor and and additive functor.

Functors of the shape $V \mapsto V^{\otimes d}$ (between module categories) are typical examples of polynomial functors of degree d.

Counterexample: if a is a non-zero object of a small additive category $\mathcal{A}$, the functor $P^a_{\mathcal{A}}$ of $\mathcal{F}(\mathcal{A}; K)$ is *not* polynomial.

Algebraic topology (I) : Eilenberg-MacLane (1954)

Let i and n be nonnegative integers, V and abelian group. The Eilenberg-MacLane space $K(V,n)$ (i.e. pointed topological space whose homotopy is V in degree n and zero elsewhere) gives rise, by taking singular homology, to functors $V \mapsto H_i(K(V,n);K)$ that we can see as objetcs of $\mathcal{F}(\mathbf{Ab}^{\mathrm{fg}};K)$ (where $\mathbf{Ab}^{\mathrm{fg}}$ is the category of finitely generated abelian groups), that Eilenberg and MacLane studied during the 1950's. One prove a lot of things about these functors, but it is very hard to give a complete description of them, except for special values of n and i.

These functors are usally not additive, but they are *polynomial*; that is why Eilenberg and MacLane introduced the notion of polynomiality of functors.

Algebraic topology (II): after Eilenberg-MacLane

Dold and Puppe ont introduced (1961) *derived functors of non-additive functors* between (nice) abelian categories by using a simplicial construction relying Dold-Kan correspondence; the homology of Eilenberg-MacLane spaces can be directly interpreted as derived functor of the linearisation functor $K[-] : \mathbf{Ab} \to K\text{-}\mathbf{Mod}$.

Curtis spectral sequence (1963, 1965) converges to homotopy groups of spheres (or more generally of Moore spaces) and has a first page given by the derived functors à la Dold-Puppe of Lie functors (some polynomial endofunctors of abelian groups related to the lower central series of free groups).
Non-additive derived functors and their application to Curtis spectral sequence gave also rise to much more recent work by Breen and Mikhailov (2011).

Algebraic topology (III) : Henn-Lannes-Schwartz (1993)

In the sequel of Lannes' work on the T functor, Henn, Lannes and Schwartz studied in the early 1990's a functor from the category $\mathcal{U}$ of *unstable modules* over the Steenrod algebra $\mathcal{A}(p)$ (where p is a given prime) of stable operations of mod p cohomology to the category $\mathcal{F}(\mathbb{F}_p, \mathbb{F}_p)$. They proved that this functor induces an equivalence between the quotient of $\mathcal{U}$ by the localising subcategory of *nilpotent* unstable modules and the full subcategory of *analytic* (i.e. colimit of polynomial subfunctors) functors of $\mathcal{F}(\mathbb{F}_p, \mathbb{F}_p)$.

That is the starting point of the systematic study of functor categories $\mathcal{F}(\mathbb{F}_p, \mathbb{F}_p)$ by algebraic topologists (L. Schwartz, N. Kuhn, G. Powell...).

Relations with representation theory

Polynomial functors give a very efficient framework to relate linear representations of symmetric groups to the ones of general linear groups. The prehistory of this relation, about which we will come back later, goes back to Schur's thesis and is yet present with a functorial point of view in Macdonald, *Symmetric functions and Hall polynomials* (1979), whereas the notion of *polynomial representation* of general linear groups where systematically studied in the book with this name published by Green in 1980.

Auslander introduced from the 1960's the use in representation theory (especially, of Artin algebras) of categories of *additive* functors between module categories.

Functor homology and homology of linear groups

In his unpublished thesis, Scorichenko (2000) proved the following striking result (obtained a few times before in the particular case of finite fields, independently by Betley and Suslin).

Theorem (Scorichenko)

Let R be a ring and $X : \mathbf{P}(R)^{\mathrm{op}} \times \mathbf{P}(R) \to \mathbf{Ab}$ a polynomial bifunctor. For each $d \in \mathbb{N}$, a natural split short exact sequence

$$0 \to \bigoplus_{i+j=d} H_i(\mathrm{GL}_\infty(R);\mathbb{Z}) \otimes_{\mathbb{Z}} HH_j(\mathbf{P}(R);X) \to H_d(\mathrm{GL}_\infty(R);X_\infty)$$

$$\to \bigoplus_{i+j=d-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(\mathrm{GL}_\infty(R);\mathbb{Z}), HH_j(\mathbf{P}(R);X)) \to 0$$

where $\mathrm{GL}_\infty(R) := \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{GL}_n(R)$, $X_\infty := \operatorname*{colim}_{n\in\mathbb{N}} X(R^n, R^n)$, and HH_* denotes Hochschild homology.

Corollary

If K is a perfect field of prime characteristic, for each polynomial bifunctor X of $\mathcal{F}(\mathbf{P}(K)^{\mathrm{op}} \times \mathbf{P}(K); K)$, there is a natural isomorphism

$$H_* * (\mathrm{GL}_\infty(K); X_\infty) \simeq HH_*(\mathbf{P}(K); X)$$

of graded K-vector spaces.

The abelian group $HH_*(\mathbf{P}(R); X)$ is computable or at least controllable for several nice rings R and reasonnable polynomial bifunctors X. It holds in particular if R is a field (in characteristic 0, it is much easier than in prime characteristic).

Abelian categories I: Grothendieck categories

Most of the abelian categories that we will deal with will be *Grothendieck categories*, that is abelian categories having a generator, in which arbitrary colimits exist and in which *filtered* colimits are exact.

In a Grothendieck category:

1. arbitrary limits exist;
2. the class of subobjects of any given object forms a set;
3. there exists an *injective cogenerator*; in particular, the category has enough injective objects.

Functor categories $\mathcal{F}(\mathcal{C}; K)$ (where $\mathcal{C}$ is an arbitrary small category) are Grothendieck categories.

Homological algebra

As a Grothendieck category $\mathcal{E}$ has enough injective objects, one can right derive any left-exact functor from $\mathcal{E}$ to an abelian category. In particular, one can define in $\mathcal{E}$ *extension groups*, getting functors $\mathrm{Ext}^n_{\mathcal{E}} : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathbf{Ab}$ ($\mathrm{Ext}^*_{\mathcal{E}}(X, -)$ is the right derived functor of $\mathrm{Hom}_{\mathcal{E}}(X, -)$).

As categories $\mathcal{F}(\mathcal{C}; K)$ have also enough *projective* objects (thanks to the $P^c_{\mathcal{C}}$), extension groups in them can also be obtained by deriving contravariant functors $\mathrm{Hom}_{\mathcal{F}(\mathcal{C};K)}(-, Y)$.
One can also define *torsion groups* $\mathrm{Tor}^{\mathcal{C}}_*(-,-)$ over a small category $\mathcal{C}$ by *left* deriving with respect to one or the other variables the *tensor product over* $\mathcal{C}$

$$- \underset{\mathcal{C}}{\otimes} - : \mathcal{F}(\mathcal{C}^{\mathrm{op}}; K) \times \mathcal{F}(\mathcal{C}; K) \to \mathbf{Ab}$$

which is characterised by its commutation to colimits with respect to each variable and a natural isomorphism $F \underset{\mathcal{C}}{\otimes} P^c_{\mathcal{C}} \simeq F(c)$.

Quotient abelian categories

A subcategory $\mathcal{C}$ of an abelian category $\mathcal{E}$ is called *thick* if it is stable under subobject, quotient and extensions. Such a subcategory gives rise to a *quotient category* $\mathcal{E}/\mathcal{C}$ (Grothendieck, Gabriel) which is obtained from $\mathcal{E}$ by formally inverting morphisms whose kernel and cokernel belong to $\mathcal{C}$. Such a quotient category exists if the class of subobjects of any given object in $\mathcal{E}$ forms a set, for example if $\mathcal{E}$ is a Grothendieck category.

The canonical functor $\mathcal{E} \to \mathcal{E}/\mathcal{C}$ is *exact*. If it has a right (resp. left, resp. on both sides) adjoint, one says $\mathcal{C}$ that $\mathcal{C}$ is *localising* (resp. *colocalising*, *bilocalising*).

Let us assume that $\mathcal{E}$ is a Grothendieck category and that $\mathcal{C}$ is a thick subcategory of $\mathcal{E}$. Then $\mathcal{C}$ is localising iff $\mathcal{C}$ is stable under arbitrary direct sums; $\mathcal{C}$ and $\mathcal{E}/\mathcal{C}$ are then Grothendieck categories.

$\mathcal{C}$ is colocalising iff it is bilocalising iff it is stable under arbitrary products.

Typical example: in the category $\mathbf{Ab}$, the full subcategory $\mathbf{Ab}_{\mathrm{tor}}$ of torsion abelian groups is localising (but not colocalising). The rationalisation functor $\mathbf{Ab} \to \mathbb{Q}\text{-}\mathbf{Mod}$ induces an equivalence $\mathbf{Ab}/\mathbf{Ab}_{\mathrm{tor}} \xrightarrow{\simeq} \mathbb{Q}\text{-}\mathbf{Mod}$.

Lattice of subobjects and finiteness properties in Grothendieck categories

Let $\mathcal{E}$ be a Grothendieck category and X an object of $\mathcal{E}$. The set $\mathrm{Sob}(X)$ of subobjects of X, ordered by inclusion, is a *complete lattice*: every subset has an upper and a lower bound.

One says that X is:

- *noetherian* if each increasing sequence of subobjects of X stops;
- *artinian* if each decreasing sequence of subobjects of X stops;
- *finite* if it is both noetherian and artinian (what is equivalent to the existence of a finite filtration whose subquotients are *simple*, i.e. with a lattice of subobjects with exactly 2 elements; the length of the filtration and the isomorphism type of the subquotients, up to the order, are uniquely determined by X);
- *finitely generated* if every family of subobjects of X whose union (i.e. upper bound for inclusion) of X has a finite subfamily whose union is X.

The full subcategory of noetherian (resp. artinian, finite) objects of $\mathcal{E}$ is thick. The subcategory of finitely generated objects is stable under quotient and extensions, but generally not by subobject. An object is noetherian iff every subobject of it is finitely generated.

The category $\mathcal{E}$ is said *locally noetherian* (resp. *locally finitely generated*, *locally finite*) is it is generated by its noetherian (resp. finitely generated, finite) objects, i.e. if each object in it is locally noetherian (resp. locally finitely generated, locally finite), i.e. is the sum of its noetherian (resp. finitely generated, finite) subobjects.

The category $\mathcal{F}(\mathcal{C};K)$ is locally finitely generated because it is generated by the finitely generated functors $P_{\mathcal{C}}^{c}$. But it is rarely locally noetherian (even if K is a field).

Proposition

If X is an object of a Grothendieck category $\mathcal{E}$, the following statements are equivalent:

1. *X is finitely generated;*
2. *the functor $\mathrm{Hom}_{\mathcal{E}}(X,-) : \mathcal{E} \to$ **Ab** commutes with filtered colimits of monomorphisms;*
3. *if I is a small filtered category and $\Phi : I \to \mathcal{E}$ a functor, then the canonical map*

$$\operatorname*{colim}_{I} \mathrm{Hom}_{\mathcal{E}}(X,-) \circ \Phi \to \mathrm{Hom}_{\mathcal{E}}(X, \operatorname*{colim}_{I} \Phi)$$

is injective.

The proof of this classical property relies on the exactness of filtered colimits in $\mathcal{E}$.

Finite presentation

Definition

One says that an object X of $\mathcal{E}$ is *finitely presented* if the functor $\mathrm{Hom}_{\mathcal{E}}(X,-)$ commutes with filtered colimits.

Unlike finite type, finite presentation does not depend only on lattices of subobjects, it is a global property in the category (for example, a *simple* object is not always finitely presented, even in a module category — the R-module R/I, where R is a commutative ring and I a maximal ideal of R which is not finitely generated).

Higher finite presentation

One defines the property of higher finite presentation of order n, where $n \in \mathbb{N} \cup \{\infty\}$ — for short fp_n — as follows.

Definition

An object X of a Grothendieck category $\mathcal{E}$ is said to have property fp_n if the functor $\mathrm{Ext}^i_{\mathcal{E}}(X, -)$ commutes with filtered colimits of monomorphisms for every integer $i \leq n$.

So, fp_0 is nothing but finite generation.

Proposition

For $n > 0$, the following statements are equivalent:

1. *X satisfies fp_n;*
2. *the functor $\mathrm{Ext}^i_{\mathcal{E}}(X, -)$ commutes with filtered colimits for $i < n$.*

In particular, fp_1 is finite presentation.

Fundamental examples

Every finitely generated *projective* object of a Grothendieck category $\mathcal{E}$ satisfies property fp_∞.

Proposition

If the category $\mathcal{E}$ is locally noetherian, then every finitely generated object of $\mathcal{E}$ satisfies fp_∞.

Conversely, if $\mathcal{E}$ is locally finitely generated and if every finitely generated object of $\mathcal{E}$ is finitely presented, then $\mathcal{E}$ is locally noetherian.

It is generally not easy to know whether a finitely generated object of $\mathcal{E}$ satisfies fp_∞ (for example, if K is a field and G an infinite group, the problem to know whether the $K[G]$-module K satisfies fp_∞ is interesting and hard).

Property fp_n and short exact sequences

The classical following property comes from formal arguments of cohomological functors.

Proposition

Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of a Grothendieck category $\mathcal{E}$ and $n \in \mathbb{N} \cup \{\infty\}$.

1. *If X and Z satisfy fp_n, then Y too.*
2. *If $X \in pf_n(\mathcal{E})$ and $Y \in pf_{n+1}(\mathcal{E})$, then $Z \in pf_{n+1}(\mathcal{E})$.*
3. *If $Z \in pf_{n+1}(\mathcal{E})$ and $Y \in pf_n(\mathcal{E})$, then $X \in pf_n(\mathcal{E})$.*

Proprerty fp_n and finitely generated projective resolutions

Corollary

Let us assume that $\mathcal{E}$ has a set of objects F which generate $\mathcal{E}$ and is included in $fp_n(\mathcal{E})$. Then an object X of $\mathcal{E}$ satisfies fp_n iff there exists an exact sequence

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

where every object P_i is a finite direct sum of elements of F.

The main case is when the elements of F are finitely generated projectives — for example, when F is the set of functors $P^c_{\mathcal{C}}$ in $\mathcal{F}(\mathcal{C}; K)$.

Applications of property fp_n (I): a finiteness property

Proposition

Let $\mathcal{C}$ be a small category, $n \in \mathbb{N}$ and F a functor of $\mathcal{F}(\mathcal{C}; K)$ satisfying fp_n. For each functor G (resp. H) of $\mathcal{F}(\mathcal{C}; K)$ (resp. $\mathcal{F}(\mathcal{C}^{\mathrm{op}}; K)$) taking values in noetherian K-modules and each integer $i \leq n$, the K-module $\mathrm{Ext}^i_{\mathcal{F}(\mathcal{C};K)}(F, G)$ (resp. $\mathrm{Tor}^{\mathcal{C}}_i(H, F)$) is noetherian.

Applications of property fp_n (II): a Künneth formula

Proposition

Let us assume that K is a field. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and F, G (resp. X, Y) functors in $\mathcal{F}(\mathcal{C};K)$ (resp. $\mathcal{F}(\mathcal{D};K)$). There exists a natural map of graded K-vector spaces

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C};K)}(F,G)\otimes_K \mathrm{Ext}^*_{\mathcal{F}(\mathcal{D};K)}(X,Y)\to \mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}\times\mathcal{D};K)}(F\boxtimes_K X, G\boxtimes_K Y)$$

which is bijective in degrees $< n$ and injective in degree n if F and X satisfy fp_{n-1}.

Here $\boxtimes_K$ denotes the *exterior tensor product* over K, given by $(F\boxtimes_K X)(c,d) = F(c)\otimes_K X(d)$.

Polynomial functors on $\mathbf{P}(R)$

Let R be a ring and $\mathcal{E}$ a Grothendieck category. One defines the *difference functor* $\Delta : \mathbf{Fun}(\mathbf{P}(R), \mathcal{E}) \to \mathbf{Fun}(\mathbf{P}(R), \mathcal{E})$ by

$$\Delta(F)(V) := \mathrm{Ker}\,(F(V \oplus R) \twoheadrightarrow F(V)) \simeq \mathrm{Coker}\,(F(V) \hookrightarrow F(V \oplus R))\ ;$$

one has so a natural splitting $F \circ (- \oplus R) \simeq F \oplus \Delta(F)$.

Definition

Let $d \in \mathbb{N}$ and $F : \mathbf{P}(R) \to \mathcal{E}$ be a functor. One says that F is polynomial of degree at most d if $\Delta^{d+1}(F) = 0$.

As the functor Δ is exact (and commutes with limits and colimits), the full subcategory $\mathcal{P}ol_d(\mathbf{P}(R), \mathcal{E})$ on polynomial functors of degree at most d is *bilocalising* in $\mathbf{Fun}(\mathbf{P}(R), \mathcal{E})$.
We will go back in the second talk to these subcategories, with a general additive source category (and a variation of the previous definition).

Support of n-presentation

Definition

Let $\mathcal{E}$ be a Grothendieck category, $\mathcal{C}$ a small category, S a set of objects of $\mathcal{C}$, $n \in \mathbb{N}$ and $F : \mathcal{C} \to \mathcal{E}$ a functor. One says that S is a *support of n-presentation* of F if there exists an exact sequence

$$X_n \to X_{n-1} \to \cdots \to X_0 \to F \to 0$$

where each functor X_i has the shape

$$\bigoplus_{s \in S} M_s[\mathcal{C}(s, -)]$$

where M_s are objetcs of $\mathcal{E}$.

($M[T]$: direct sum of copies of M labelled by T.)

Property pfs_n

One says that a functor satisfies property pfs_n (where $n \in \mathbb{N}$) if it has a finite support of n-presentation. One says that a functor satisfies property pfs_∞ if it satisfies pfs_n for all $n \in \mathbb{N}$.

Property pfs_n is a "relative" form of property fp_n — we are going to make this precise now.

Relations between proprerties fp_n and pfs_n

Let $\mathcal{C}$ be a small category, $\mathcal{E}$ a Grothendieck category and $n \in \mathbb{N} \cup \{\infty\}$.

Proposition

Every functor of $\mathbf{Fun}(\mathcal{C}, \mathcal{E})$ *satisfying* pf_n *satisfies also* pfs_n.

Proposition

Let us assume that, for all objects a and b of $\mathcal{C}$, *the set* $\mathrm{Hom}_{\mathcal{C}}(a, b)$ *is finite. Then the following statements are equivalent for a functor F of* $\mathbf{Fun}(\mathcal{C}, \mathcal{E})$.

1. *F satifies property* fp_n.
2. *F satifies property* pfs_n *and its values belong to* $pf_n(\mathcal{E})$.

Property pfs_n and difference functor

We begin by introducing a quantitative version of ther property pfs_n for each functor $F : \mathbf{P}(R) \to \mathcal{E}$: one says that F satisfies $pfs_n(m)$, where $m \in \mathbb{N}$, if $\{R^m\}$ is a support of n-presentation of F. It is the same to require that $\{R^i \mid i \leq n\}$ is a support of m-presentation of F.

Proposition

1. *If F satisfies $psf_n(m)$, then the same is true for $\Delta(F)$.*
2. *If $\Delta(F)$ satisfies $psf_n(m)$, then F satisfies $psf_n(n+m+1)$.*

Schwartz Lemma (version pfs_∞)

Theorem

Let R be a ring, $\mathcal{E}$ a Grothendieck category and $n, d \in \mathbb{N}$. Then each functor of $\mathcal{P}ol_d(\mathbf{P}(R), \mathcal{E})$ satisfies property $psf_n(d.(n+1))$.
In particular, each polynomial functor of $\mathbf{Fun}(\mathbf{P}(R), \mathcal{E})$ satisfies pfs_∞.

Schwartz Lemma (case of a finite ring at the source)

Corollary

Let R be a **finite** *ring and F a polynomial functor of $\mathcal{F}(R, K)$. Let us assume that the ring K is noetherian. The following statements are equivalent:*

1. *F is finitely generated;*
2. *F takes values in finitely generated K-modules;*
3. *F satisfies pf_∞.*

The equivalence between 1 and 3 is false for an arbitrary ring; we will nevertheless see that it remains true if R is a commutative finitely generated ring, but with a quite different proof.

Postlude: Putman-Sam-Snowden Noetherianity Theorem

In fact, the equivalence of the statements of the previous theorem remains true for an arbitrary (non-polynomial) functor of $\mathcal{F}(R, K)$, thanks to the following result:

Theorem (Putman-Sam-Snowden)

If the ring R is finite and the ring K is noetherian, then the category $\mathcal{F}(R, K)$ is locally noetherian.

Nevertheless, this result is about 20 years later than Schwartz Lemma and is quite harder to prove.
Moreover, if R is infinite, the category $\mathcal{F}(R, K)$ is *not* locally noetherian, whereas property fp_∞ remains true for wide classes of polynomial functors.

(Temporary) end

Thank you for your attention!

Cảm ơn