Noetherianity and homological finiteness of polynomial functors (II)

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Cross-effects

In this whole talk, $\mathcal{A}$ denotes an essentially small additive category, $\mathcal{E}$ an abelian category, R a ring and k a commutative ring.
We are especially interested with the case $\mathcal{A} = \mathbf{P}(R)$ (finitely generated projective left R-modules) and $\mathcal{E} = k$-**Mod**.

Definition (Eilenberg-MacLane)

Let $F : \mathcal{A} \to \mathcal{E}$ be a functor and $d \geq 0$ an integer. One defines the *d-th cross-effect* of F as the functor $cr_d(F) : \mathcal{A}^d \to \mathcal{E}$ given by

$$cr_d(F)(a_1, \dots, a_d) = \operatorname{Ker}\left(F\left(\bigoplus_{i=1}^{d} a_i\right) \to \bigoplus_{i=1}^{d}\left(F\left(\bigoplus_{j \neq i} a_j\right)\right)\right)$$

where morphisms are induced by the canonical projections

$$\bigoplus_{i=1}^{d} a_i \twoheadrightarrow \bigoplus_{j \neq i} a_j.$$

One has a natural splitting

$$F\Big(\bigoplus_{i=1}^{d} a_i\Big) \simeq \bigoplus_{1\le i_1<\cdots<i_r\le n} cr_r(F)(a_{i_1},\dots,a_{i_r}).$$

So, the functor $cr_d : \mathbf{Fun}(\mathcal{A},\mathcal{E}) \to \mathbf{Fun}(\mathcal{A}^d,\mathcal{E})$ is a *direct summand* of precomposition by the d-th iterated direct sum $\mathcal{A}^d \to \mathcal{A}$, and in particular, cr_d commutes to limits and colimits (so, it is exact).

Definition (Eilenberg-MacLane)

A functor $F : \mathcal{A} \to \mathcal{E}$ is *polynomial* of degree at most d if $cr_{d+1}(F) = 0$. We denote by $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})$ the full subcategory of $\mathbf{Fun}(\mathcal{A},\mathcal{E})$ on these functors.

(It is easy to check that, for $\mathcal{A} = \mathbf{P}(R)$, this definition is equivalent to the one given in the first talk.)

An easy finiteness proprerty

The following property can be easily deduced from the natural splitting of the previous slide.

Proposition

For each $d \in \mathbb{N}$, the evaluation functor on R^d induces an exact and faithful functor $\mathcal{P}ol_d(\mathbf{P}(R), \mathcal{E}) \to \mathcal{E}$.

So, if F belongs to $\mathcal{P}ol_d(\mathbf{P}(R), \mathcal{E})$, the lattice of subfunctors of F is a sub-ordered set of the lattice of subfunctors of $F(R^d)$ in $\mathcal{E}$, whence:

Corollary

If $F : \mathbf{P}(R) \to \mathcal{E}$ is a polynomial functor whose values are noetherian (resp. artinian, finite) in $\mathcal{E}$, then F is noetherian (resp. artinian, finite) in $\mathbf{Fun}(\mathbf{P}(R), \mathcal{E})$

Two applications

If the ring R is finite, each finitely-generated functor (not assumed to be polynomial) of $\mathcal{F}(R, k)$ has values in finitely-generated k-modules, so :

Corollary

Si R is a finite ring and k is a field, then each polynomial functor of $\mathcal{F}(R, k)$ is locally finite.

If the underlying additive group of R is finitely-generated, each finitely-generated polynomial functor of $\mathcal{F}(R, k)$ has values in finitely-generated k-modules, whence:

Corollary

If the underlying additive group of R is finitely-generated and if the ring k is artinian (resp. noetherian), then each polynomial functor of $\mathcal{F}(R, k)$ is locally finite (resp. locally noetherian).

Pirashvili's recollement Theorem (preliminaries)

As functors cr_n commute with limits and colimits, the subcategories $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})$ are *bilocalising* in $\mathbf{Fun}(\mathcal{A},\mathcal{E})$ (at least if $\mathcal{E}$ is a Grothendieck category). One can look at the abelian quotient categories $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})/\mathcal{P}ol_{d-1}(\mathcal{A},\mathcal{E})$. Pirashvili described in the late 1980's these categories.

For F in $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})$, the multifunctor $cr_d(F):\mathcal{A}^d\to\mathcal{E}$ is *additive* with respect to each of its d entries. Moreover, it is endowed, for each permutation $\sigma\in\mathfrak{S}_d$, with natural isomorphisms

$$cr_d(F)(a_{\sigma(1)},\dots,a_{\sigma(d)})\simeq cr_d(F)(a_1,\dots,a_d)$$

compatible in a suitable sense with composition of permutations. One says that $cr_d(F)$ is a *symmetric* d-multifunctor on $\mathcal{A}$ (with values in $\mathcal{E}$).

Pirashvili's recollement Theorem (statement)

We denote by $\mathbf{Add}_d(\mathcal{A},\mathcal{E})$ the category of multifunctors $\mathcal{A}^d \to \mathcal{E}$ which are additive with respect to each entry (it is a full subcategory of $\mathbf{Fun}(\mathcal{A}^d,\mathcal{E})$), and by $\Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})$ the category of *symmetric* multifunctors of $\mathbf{Add}_d(\mathcal{A},\mathcal{E})$. Morphisms in $\Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})$ are natural transformations that commute with symmetry isomorphisms (which are part of the structure). So, cr_d induces an exact functor $\mathcal{P}ol_d(\mathcal{A},\mathcal{E}) \to \Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})$. Its kernel is by definition $\mathcal{P}ol_{d-1}(\mathcal{A},\mathcal{E})$, so it induces an exact and faithful functor $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})/\mathcal{P}ol_{d-1}(\mathcal{A},\mathcal{E}) \to \Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})$.

Theorem (Pirashvili)

The functor cr_d induces an equivalence of categories

$$\mathcal{P}ol_d(\mathcal{A},\mathcal{E})/\mathcal{P}ol_{d-1}(\mathcal{A},\mathcal{E}) \xrightarrow{\simeq} \Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})\,.$$

Pirashvili's recollement Theorem (fundamental special case)

The category $\mathbf{Add}_d(\mathbf{P}(R), k\text{-}\mathbf{Mod})$ is equivalent to $(k \otimes (R^{\mathrm{op}})^{\otimes d})\text{-}\mathbf{Mod}$, where tensor products are taken ove $\mathbb{Z}$.

In general, if a group G acts on a commutative ring K, let us denote by $K \rtimes G$ the *twisted group algebra* of G over K: its underlying K-module is the same as the usual group algebra $K[G]$, and multiplication is given by

$$(\lambda[g]).(\mu[h]) = (\lambda(g_*\mu))[gh]$$

(for $(\lambda, \mu, g, h) \in K \times K \times G \times G$).

Then

$$\Sigma\mathbf{Add}_d(\mathbf{P}(R), k\text{-}\mathbf{Mod}) \simeq (k \otimes (R^{\mathrm{op}})^{\otimes d}) \rtimes \mathfrak{S}_d\text{-}\mathbf{Mod}\,,$$

where $\mathfrak{S}_d$ acts on $k \otimes (R^{\mathrm{op}})^{\otimes d}$ by permuting factors of the tensor product.

Pirashvili's recollement Theorem (elements of proof)

The functor $cr_d : \mathbf{Fun}(\mathcal{A}, \mathcal{E}) \to \mathbf{Fun}(\mathcal{A}^d, \mathcal{E})$ takes values in the full subcategory $\mathbf{Fun}^{d-\mathrm{red}}(\mathcal{A}^d, \mathcal{E})$ on *d-reduced* multifunctors, meaning that is it zero on each d-tuple of objects of $\mathcal{A}$ whose at least one component is zero.

Proposition (sum/diagonal adjunction)

The functor $\mathbf{Fun}(\mathcal{A}, \mathcal{E}) \to \mathbf{Fun}^{d-\mathrm{red}}(\mathcal{A}^d, \mathcal{E})$ induced by cr_d is adjoint on both sides to restriction to $\mathbf{Fun}^{d-\mathrm{red}}(\mathcal{A}^d, \mathcal{E})$ of precomposition δ_d^* by d-iterated diagonal $\delta_d : \mathcal{A} \to \mathcal{A}^d$.
This adjunction restricts to the functor $\mathcal{P}ol_d(\mathcal{A}, \mathcal{E}) \to \mathbf{Add}_d(\mathcal{A}, \mathcal{E})$ induced by cr_d, which is adjoint on both sides to the functor $\Delta_d : \mathbf{Add}_d(\mathcal{A}, \mathcal{E}) \to \mathcal{P}ol_d(\mathcal{A}, \mathcal{E})$ induced by δ_d^*.

If X is an object of $\Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})$, the symmetric structure on X induces an action of the group $\mathfrak{S}_d$ on the functor $\Delta_d(X)$ of $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})$.

Proposition

The functor $\mathcal{P}ol_d(\mathcal{A},\mathcal{E}) \to \Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})$ induced by cr_d is left adjoint to the functor

$$\Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E}) \to \mathcal{P}ol_d(\mathcal{A},\mathcal{E}) \qquad X \mapsto \Delta_d(X)^{\mathfrak{S}_d}$$

(invariants under the action of $\mathfrak{S}_d$).

If F is a functor of $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})$, one checks easily that the kernel and the cokernel of the unit of the adjunction $F \to \Delta_d cr_d(F)^{\mathfrak{S}_d}$ belong to $\mathcal{P}ol_{d-1}(\mathcal{A},\mathcal{E})$.
As a consequence, the functor $(\Delta_d cr_d)^{\mathfrak{S}_d}$ induces identity on $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})/\mathcal{P}ol_{d-1}(\mathcal{A},\mathcal{E})$.
On checks also that the composite functor $cr_d(\Delta_d^{\mathfrak{S}_d})$ is isomorphic to the identity of $\Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})$.

Divided powers

If V is a k-module, one defines the algebra of *divided powers* of V over k, denoted by $\Gamma^*_k(V)$, as the associative, commutative and unital k-algebra generated by elements $v^{[n]}$ for $v \in V$ and $n \in \mathbb{N}$ with the following relations:

- $\forall v \in V, \qquad v^{[0]} = 1$;
- $\forall (\lambda, v, n) \in k \times V \times \mathbb{N}, \qquad (\lambda v)^{[n]} = \lambda^n . v^{[n]}$;
- $\forall (v, n, m) \in V \times \mathbb{N} \times \mathbb{N}, \qquad v^{[n]}.v^{[m]} = \frac{(n+m)!}{n!m!}.v^{[n+m]}$;
- $\forall (v, w, n) \in V \times V \times \mathbb{N}, \qquad (v+w)^{[n]} = \sum_{i+j=n} v^{[i]}.w^{[j]}$.

This algebra is graded by $\deg(v^{[n]}) := n$. One denotes by $\Gamma^d_k(V)$ the d-th homogeneous component of the graded algebra $\Gamma^*_k(V)$. One defines so an endofunctor Γ^d_k of k-modules called d-th divided power.

The functor Γ^d_k is polynomial of degree d. It preserves surjective morphisms and filtered colimits, and one has a natural k-linear morphism $\Gamma^d_k(V) \to (V^{\otimes d})^{\mathfrak{S}_d}$ which is an isomorphism if V is a *flat* module.

The functor Γ^d_k is compatible base-change of the ring k in the following sense: if $k \to K$ is a ring morphism, with K commutative, one has a K-linear isomorphism $\Gamma^d_K(K \otimes_k V) \simeq K \otimes_k \Gamma^d_k(V)$, natural in the k-module V.

The functor Γ^d_k has a canonical structure of *symmetric monoidal* endofunctor of the symmetric monoidal category $(k\text{-}\mathbf{Mod}, \otimes, k)$. This allows to lift Γ^d_k as an endofunctor of k-algebras.

Strict polynomial functors (à la Friedlander-Suslin)

If $\mathcal{A}$ is a k-linear category, one defines a k-linear category $\Gamma^d_k\mathcal{A}$ with the same objects as $\mathcal{A}$ and with morphisms given by $(\Gamma^d_k\mathcal{A})(a,b) := \Gamma^d_k(\mathcal{A}(a,b))$ and composition by

$$\Gamma^d_k(\mathcal{A}(a,b)) \otimes_k \Gamma^d_k(\mathcal{A}(b,c)) \to \Gamma^d_k(\mathcal{A}(a,b) \otimes_k \mathcal{A}(b,c)) \to \Gamma^d_k(\mathcal{A}(a,c))$$

where the first map is induced by the monoidal structure of Γ^d_k and the seconde one is induced by composition in $\mathcal{A}$.

Definition

Let $\mathcal{A}$ a small k-linear additive category, $\mathcal{E}$ be a k-linear abelian category and $d \in \mathbb{N}$. We denote by $\mathcal{P}_{d;k}(\mathcal{A},\mathcal{E})$ the category $\mathbf{Add}_k(\Gamma^d_k(\mathcal{A}),\mathcal{E})$ of k-linear functors $\Gamma^d_k(\mathcal{A}) \to \mathcal{E}$. Its objects are called homogeneous *strict polynomial functors* of degree d over k from $\mathcal{A}$ to $\mathcal{E}$.
One writes $\mathcal{P}_{d;k}(\mathcal{A};k)$ for $\mathcal{P}_{d;k}(\mathcal{A}, k\text{-}\mathbf{Mod})$; when R and S are k-algebras, one writes $\mathcal{P}_{d;k}(R,S)$ for $\mathcal{P}_{d;k}(\mathbf{P}(R), S\text{-}\mathbf{Mod})$.

This notion was introduced, in the case of $\mathcal{P}_{d;k}(k,k)$, by Friedlander and Suslin (*Inventiones* 1997). It is related to representations of *algebraic* linear groups.

Relation with ordinary polynomial functors

There is a canonical k-linear morphism $k[V] \to \Gamma^d_k(V) \quad v \mapsto v^{[d]}$ (natural in a k-module V). It is compatible with symmetric monoidal structures of functors $k[-]$ and Γ^d_k. As a consequence, if $\mathcal{A}$ is a k-linear category, one gets a k-linear functor $k[\mathcal{A}] \to \Gamma^d_k(\mathcal{A})$ which is identity on objects.

Let us now assume that $\mathcal{A}$ is k-linear, additive and small. The functor $\mathcal{P}_{d;k}(\mathcal{A}, \mathcal{E}) \to \mathbf{Fun}(\mathcal{A}, \mathcal{E})$ (where $\mathcal{E}$ is a k-linear abelian category) obtained by precomposition by the previous functor takes values in $\mathcal{P}ol_d(\mathcal{A}, \mathcal{E})$.
So, one gets a functor $i_d : \mathcal{P}_{d;k}(\mathcal{A}, \mathcal{E}) \to \mathcal{P}ol_d(\mathcal{A}, \mathcal{E})$ which is exact (it commutes even with all limits and colimits) and faithful. It is nevertheless generally not full.

Proposition

A functor F of $\mathcal{P}_{d;k}(\mathcal{A},\mathcal{E})$ is finitely generated if and only if the functor $i_d(F)$ of $\mathcal{P}ol_d(\mathcal{A},\mathcal{E})$ is finitely generated.

Let us now assume that $k=\mathbb{Z}$. The natural morphism $\Gamma^d_{\mathbb{Z}}(V)\to V^{\otimes d}$ induces an additive functor $\Gamma^d_{\mathbb{Z}}(\mathcal{A})\to\mathcal{A}^{\otimes d}$ (the tensor product of two preadditive categories $\mathcal{A}$ and $\mathcal{B}$ is the preadditive category $\mathcal{A}\otimes\mathcal{B}$ such that $\mathrm{Ob}(\mathcal{A}\otimes\mathcal{B})=\mathrm{Ob}(\mathcal{A})\times\mathrm{Ob}(\mathcal{B})$ and $(\mathcal{A}\otimes\mathcal{B})((a,b),(a',b'))=\mathcal{A}(a,a')\otimes\mathcal{B}(b,b')$). By precomposition, we get an exact and faithful functor

$$\mathbf{Add}_d(\mathcal{A},\mathcal{E})\simeq\mathbf{Add}(\mathcal{A}^{\otimes d},\mathcal{E})\to\mathcal{P}_{d;\mathbb{Z}}(\mathcal{A},\mathcal{E})$$

whence, by precomposing with the forgetful functor, a functor

$$\tilde{\Delta}_d:\Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})\to\mathbf{Add}_d(\mathcal{A},\mathcal{E})\to\mathcal{P}_{d;\mathbb{Z}}(\mathcal{A},\mathcal{E})$$

such that $\Delta_d\simeq i_d\circ\tilde{\Delta}_d:\Sigma\mathbf{Add}_d(\mathcal{A},\mathcal{E})\to\mathcal{P}ol_d(\mathcal{A},\mathcal{E})$.

Relation with Schur algebras

The following result is a consequence of Gabriel-Popescu Theorem.

Proposition

Let R and S be k-algebras and $n \geq d \geq 0$ be integers. The evaluation functor on R^n induces a k-linear equivalence of categories

$$\mathcal{P}_{d;k}(R,S) \xrightarrow{\simeq} (\Gamma^d_k(\mathcal{M}_n(R)) \otimes_k S)\text{-}\mathbf{Mod}.$$

Algebras $\Gamma^d_k(\mathcal{M}_n(R))$ are generalisations of classical Schur algebras.

(One can also prove, from Gabriel-Popescu, that the category $\mathcal{P}ol_d(\mathbf{P}(R), S)$ is equivalent to module category, but over a ring which is not very tractable.)

A classical theorem of E. Noether

Theorem (Noether)

Let k be a commutative noetherian ring, A a finitely-generated commutative k-algebra and G a finite group acting on A (by k-algebra automorphisms). Then the algebra A^G of invariants is a finitely-generated k-algebra (in particuliar, it is noetherian).

By applying this result to a square-zero extension, one obtains:

Corollary

Under the previous assumptions, if V is a finitely-generated A-module endowed with an action of G such that $g_*(a.v) = (g_* a).(g_* v)$ for all $(g, a, v) \in G \times A \times V$, then V^G is a finitely-generated A^G-module.

A noetherian theorem for strict polynomial functors

Theorem (D.-Touzé)

Let k be a commutative noetherian ring and R a commutative k-algebra which is essentially finitely-generated (i.e. a localisation of a finitely-generated commutative k-algebra).
Then for each $d \in \mathbb{N}$, the category of strict polynomial functors $\mathcal{P}_{d;k}(R, k)$ is locally noetherian.

One can also replace R by a (non-commutative) *finite* (i.e. finitely generated as a module) algebra over a commutative essentially finitely-generated k-algebra (with the same proof).

Proof (1): reductions

1. It is the same to prove that the k-algebra $\Gamma_k^d(\mathcal{M}_d(R))$ is noetherian.
2. It is enough to prove the result when R is a finitely-generated k-algebra. (Everything commutes with localisation, and a localisation of a noetherian ring is noetherian.)
3. It is enough to prove the result when R is a finitely-generated *flat* k-algebra. In fact, any finitely-generated k-algebra is a quotient of a polynomial algebra $k[x_1, \dots, x_n]$, which k-flat, Γ_k^d preserves surjective morphisms, and any quotient of a noetherian ring is noetherian.

Proof (2): case when R is flat and finitely generated

$R^{\otimes d}$ is a finitely-generated k-algebra, and $V := (\mathcal{M}_d(R))^{\otimes d}$ is a finitely-generated $R^{\otimes d}$-module. The group $\mathfrak{S}_d$ acts on $R^{\otimes d}$ and V (is a compatible way), so that Noether Theorem implies that $(R^{\otimes d})^{\mathfrak{S}_d}$ is a finitely-generated k-algebra (so is noetherian), and that $V^{\mathfrak{S}_d}$ is a finitely-generated $(R^{\otimes d})^{\mathfrak{S}_d}$-module.

As the k-modules R and $\mathcal{M}_d(R)$ are flat, $\Gamma_k^d(R) \simeq (R^{\otimes d})^{\mathfrak{S}_d}$ and $\Gamma_k^d(\mathcal{M}_d(R)) \simeq V^{\mathfrak{S}_d}$. So, $\Gamma_k^d(\mathcal{M}_d(R))$ is a finite algebra over a finitely-generated k-algebra, implying that it is noetherian.

(Temporary) end

Thank you for your attention!

Cảm ơn