Tensor decompositions à la Steinberg for representations of additive categories

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Online talk for the second meeting of ANR AlMaRe
http://massuyea.perso.math.cnrs.fr/meeting2.html

Report on a joint work with Antoine Touzé and Christine Vespa
https://hal.archives-ouvertes.fr/hal-02103934

Original french version of these notes:
https://math.univ-lille1.fr/~djament/expose_AlMaRe_avril2021.pdf

A first theorem due to R. Steinberg

Theorem (R. Steinberg)

Let $n \geq 3$ be an integer. A finite-dimensional complex representation of the group $\mathrm{SL}_n(\mathbb{Z})$ is irreducible if and only if it is isomorphic to the tensor product of an irreducible representation factorising through the morphism $\mathrm{SL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_n(\mathbb{Z}/i)$ induced by reduction modulo an integer $i > 0$ and of an irreducible polynomial representation (that is: whose action is given by polynomials in the coefficients of matrices) of $\mathrm{SL}_n(\mathbb{Z})$. Moreover, two such irreducible representations are isomorphic if and only if their tensor decompositions are isomorphic.

Steinberg tensor product Theorem

Let p be a prime number, $q = p^r$ a power of p and $n \in \mathbb{N}$. One can associate to each p-restricted (that is: whose each part is repeated no more than $p-1$ times) partition whose parts are $\leq n-1$ an irreducible representation of the algebraic group SL_n in characteristic p. The representations of the finite groups $\mathrm{SL}_n(\mathbb{F}_q)$ obtained by taking points over the field $\mathbb{F}_q$ are called *elementary representations*.

(Another definition, in terms of functors, of elementary representations will be given later.)

Steinberg tensor product Theorem (2)

Theorem (R. Steinberg)

A representation of the finite group $\mathrm{SL}_n(\mathbb{F}_q)$ over a field containing $\mathbb{F}_q$ is irreducible if and only if it is isomorphic to a tensor product

$$M_0 \otimes M_1^{(1)} \otimes \cdots \otimes M_{r-1}^{(r-1)}$$

where M_i are elementary representations and the exponent $^{(i)}$ denotes i-th iteration of Frobenius twist.
Moreover, two such representations are isomorphic if and only if their decompositions are isomorphic.

Functor categories

Let $\mathcal{C}$ be an (essentially) small category and K be a field. We denote by $\mathcal{F}(\mathcal{C};K)$ the category of functors from $\mathcal{C}$ to K-vector spaces. One may think of it as the category of representations over K of $\mathcal{C}$ (which may be thought as a monoid with several objects).
The category $\mathcal{F}(\mathcal{C};K)$ is a nice abelian category: it has enough projectives and injectives and arbitrary (co)limits. It is a Grothendieck category.

We will be interested with the case when the source category is *additive*. A fundamental instance is the category $\mathrm{P}(R)$ of finitely-generated projective left modules over a ring R.

Non-additive representations of additive categories

Let $\mathcal{A}$ be an (essentially) small additive category. One is often interested with the category $\mathrm{Add}(\mathcal{A};K)$ of *additive* functors from $\mathcal{A}$ to K-vector spaces, especially in representation theory, from Auslander's work. It is also a nice abelian category.

$\mathrm{Add}(\mathcal{A};K)$ is a thick (that is: stable under subquotients and extensions) subcategory of $\mathcal{F}(\mathcal{A};K)$, which is stable under all limits and colimits.

Nevertheless, $\mathcal{F}(\mathcal{A};K)$ is generally much harder to understand than $\mathrm{Add}(\mathcal{A};K)$.

For example, $\mathrm{Add}(\mathrm{P}(R);K)$ is equivalent to the category of (K,R)-bimodules, but the structure of $\mathcal{F}(\mathrm{P}(R);K)$ remains widely unknown when $R=K$ is a finite field.

An historical motivation

Let i and n be natural numbers and V an abelian group. The Eilenberg-MacLane space $K(V,n)$ (that is: a pointed topological space whose homology is V in degree n and zero elsewhere) gives rise, by taking singular homology, to functors $V \mapsto H_i(K(V,n);K)$ which one may see as objects of $\mathcal{F}(\mathrm{Ab}^{\mathrm{fg}};K)$ (where $\mathrm{Ab}^{\mathrm{fg}}$ is the category of finitely generated abelian groups), which were studied by Eilenberg and MacLane during the 1950's.
One can say a lot of things about these functors, but it is very hard to given a complete description of them except when n and i are very small. These functors are generally not additive, but they have a fundamental property extending additivity, which was introduced by Eilenberg and MacLane: *polynomiality*.

Polynomial functions

One begins (following Eilenberg-MacLane) by defining polynomial functions between two abelian groups U and V. For $d \in \mathbb{N}$, one defines the d-th *deviation* of a (set-)function $f : U \to V$ as the function $\mathrm{dev}_d(f) : U^d \to V$ given by

$$\mathrm{dev}_d(f)(x_1, \ldots, x_d) := \sum_{I \subset \{1,\ldots,d\}} (-1)^{d-\mathrm{Card}(I)} f(x_I)\,,$$

where

$$x_I := \sum_{i \in I} x_i.$$

One says that f is *polynomial* of degree $\leq d$ if $\mathrm{dev}_{d+1}(f)$ is the zero function.

(Hom-)polynomial functors

Let $\mathcal{A}$ and $\mathcal{E}$ be additive categories and $F : \mathcal{A} \to \mathcal{E}$ be a functor. One says that F is Hom-*polynomial* if, for all objects x and y in $\mathcal{A}$, the function

$$F_{x,y} : \mathcal{A}(x, y) \to \mathcal{E}(F(x), F(y))$$

giving the effect of F on morphisms is polynomial.

If all functions $F_{x,y}$ are polynomial of degree $\leq d$, one says that the functor F is *polynomial* of degree $\leq d$. (Only this last notion is classical.)

Hom-polynomial functors are generally much harder to study than polynomial functors. A Hom-polynomial functor of finite length is always polynomial.

Example : elementary functors

For each $d \in \mathbb{N}$, the d-th tensor power $T^d : V \mapsto V^{\otimes d}$ (where the tensor product is taken over K) defines an endofunctor of K-vector spaces which is polynomial of degree d. The symmetric group $\mathfrak{S}_d$ acts on T^d. One names *elementary functor* (over K) each endofunctor of K-vector spaces of the shape

$$\mathrm{Im}\,(T^d \otimes M)_{\mathfrak{S}_d} \to (T^d \otimes M)^{\mathfrak{S}_d}$$

(the map being the norm), where M a K-linear irreducible representation of $\mathfrak{S}_d$. Such a functor is polynomial of degree d.

Example

The d-th exterior power Λ^d is an elementary functor of degree d.

Ideals of an additive category

An *ideal* of an additive category $\mathcal{A}$ is a subfunctor of $\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$.

Example

If R is a ring, the ideals of the additive category $\mathsf{P}(R)$ identify with two-sided ideals of R (the ideal in the usual sense is obtained by evaluating the ideal in the categorical sense on (R, R)).

If $\mathcal{I}$ is an ideal of $\mathcal{A}$, one may build a category $\mathcal{A}/\mathcal{I}$ with the same objects as $\mathcal{A}$, and morphisms $(\mathcal{A}/\mathcal{I})(x, y) := \mathcal{A}(x, y)/\mathcal{I}(x, y)$. One has a canonical additive functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ which is identity on objects.

K-cotrivial ideals

An additive category $\mathcal{A}$ is called *K-trivial* if, for all objects x and y of $\mathcal{A}$, the following conditions are fulfiled:

1. the abelian group $\mathcal{A}(x, y)$ is finite;
2. the tensor product $\mathcal{A}(x, y) \otimes_{\mathbb{Z}} K$ is zero (that is: the order of $\mathcal{A}(x, y)$ is invertible in K).

The second condition implies that the only Hom-polynomial functors of $\mathcal{F}(\mathcal{A}; K)$ are constant functors.

An ideal $\mathcal{I}$ of an additive category $\mathcal{A}$ is called *K-cotrivial* if the category $\mathcal{A}/\mathcal{I}$ is K-trivial.

Antipolynomial functors

Definition

A functor F of $\mathcal{F}(\mathcal{A}; K)$ is called *antipolynomial* if there is a K-cotrivial ideal $\mathcal{I}$ of $\mathcal{A}$ such that F factorises through the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$.

Antipolynomial functors form a class of functors which may be thought as "orthogonal" to the one of (Hom-)polynomial functors.

Global decomposition (general form)

One denotes by $\mathcal{F}^{\mathrm{fd}}(\mathcal{A};K)$ the full subcategory of functors of $\mathcal{F}(\mathcal{A};K)$ whose values are finite-dimensional K-vector spaces.

Theorem

Let F be a finitely-generated functor of $\mathcal{F}^{\mathrm{fd}}(\mathcal{A};K)$. There is a functor B of $\mathcal{F}(\mathcal{A}\times\mathcal{A};K)$, unique up to isomorphism, such that:

1. *F is isomorphic to the composite of the diagonal $\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ and B;*
2. *the bifunctor B is Hom-polynomial with respect to the first variable: for each object x of $\mathcal{A}$, the functor $B(-,x)$ of $\mathcal{F}(\mathcal{A};K)$ is Hom-polynomial;*
3. *there is a K-cotrivial ideal $\mathcal{I}$ of $\mathcal{A}$ such that B factorises through the canonical additive functor $\mathcal{A}\times\mathcal{A}\to\mathcal{A}\times\mathcal{A}/\mathcal{I}$ (in particular, B is antipolynomial with respect to the second variable).*

Global decomposition (case of simple functors)

Theorem

Let F be a simple functor of $\mathcal{F}^{\mathrm{fd}}(\mathcal{A};K)$. There is a functor B of $\mathcal{F}(\mathcal{A}\times\mathcal{A};K)$, unique up to isomorphism, such that:

1. *F is isomorphic to the composite of the diagonal $\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ and B;*
2. *the bifunctor B is polynomial with respect to the first variable;*
3. *there is a K-cotrivial ideal $\mathcal{I}$ of $\mathcal{A}$ such that B factorises through the canonical functor $\mathcal{A}\times\mathcal{A}\to\mathcal{A}\times\mathcal{A}/\mathcal{I}$.*

Moreover, B is simple, and F and B have the same endomorphism (skew-)field.

Conversally, if B is a simple bifunctor fulfilling conditions 2 and 3 above, its precomposition by the diagonal of $\mathcal{A}$ is a simple functor of $\mathcal{F}(\mathcal{A};K)$.

Case of a big enough field

Corollary

Let us assume that K contains all roots of unit. Then a functor of $\mathcal{F}^{\mathrm{fd}}(\mathcal{A}; K)$ is simple if and only if it is isomorphic to the tensor product of a simple antipolynomial functor and of a simple polynomial functor. Moreover, the decomposition is unique up to isomorphism.

This corollary is similar to the first theorem of Steinberg that we mentioned; the proofs of both statements have numerous similarities. Nevertheless, their relations are not completely understood.

Splitting field of an additive category

Definition

A field K is called a **splitting field** of the (essentially) small additive category $\mathcal{A}$ if the endomorphism (skew-)field of each simple functor with finite-dimensional values of $\mathrm{Add}(\mathcal{A};K)$ is reduced to K.

If R is a ring, K is a splitting field of the category $\mathrm{P}(R)$ if and only if it is a splitting field of the K-algabra $R^{\mathrm{op}} \otimes_{\mathbb{Z}} K$ in the usual sense of representation theory.

Example

If K is algebraically closed, it is a splitting field of $\mathcal{A}$.

Tensor decomposition of simple polynomial functors

Theorem

Let us assume that K is a splitting field of $\mathcal{A}$. Then a polynomial functor of $\mathcal{F}^{\mathrm{fd}}(\mathcal{A};K)$ is simple if and only if it is isomorphic to a tensor product

$$\bigotimes_{\pi} E_\pi \circ \pi$$

labelled by a complete set of representatives of isomorphism classes of simple functors with finite-dimensional values π of $\mathrm{Add}(\mathcal{A};K)$, where E_π are elementary endofunctors of K-vector spaces, all but a finite number isomorphic to the constant functor K.
Moreover, the functors E_π appearing in this decomposition are unique up to isomorphism.

Example: a theorem of N. Kuhn (1)

Let p be a prime number, $r > 0$ be an integer and $q := p^r$. The finite field $\mathbb{F}_q$ is a splitting field of $\mathrm{P}(\mathbb{F}_q)$ and, thanks to Galois theory of finite fields, a complete set of representatives of isomorphism classes of simple $(\mathbb{F}_q, \mathbb{F}_q)$-bimodules is given by ${}^{(i)}\mathbb{F}_q$ for $0 \leq i < r$, meaning $\mathbb{F}_q$ endowed with the obvious structure of left $\mathbb{F}_q$-vector space, and of the structure of right $\mathbb{F}_q$-vector space obtained by twisting the obvious one by the i-th iteration of the Frobenius morphism $x \mapsto x^p$.

Example: a theorem of N. Kuhn (2)

Le previous theorem says us then that the simple objects of $\mathcal{F}(\mathrm{P}(\mathbb{F}_q);\mathbb{F}_q)$ are exactly the tensor products $E_0 \otimes E_1^{(1)} \otimes \cdots \otimes E_{r-1}^{(r-1)}$, where E_i are elementary functors and the exponent ${}^{(i)}$ denotes precomposition by the i-ème iteration of the Frobenius twist functor (that is: tensorisation with ${}^{(i)}\mathbb{F}_q$).

This result is an analogue of Steinberg's tensor product Theorem. It was already gotten by Kuhn (with different methods).

Examples of applications: tensor products (1)

The following statement is easily deduced from our theorem of polynomial decomposition à la Steinberg. No assumption on the field K is needed.

Corollary

The class of polynomial functors of finite length of $\mathcal{F}^{\mathrm{fd}}(\mathcal{A};K)$ *is stable under tensor product.*

Examples of applications: tensor products (2)

The statement below uses *both* decomposition theorems à la Steinberg that we gave. To avoid a technical statement, we deal only with a source category of the shape $\mathrm{P}(R)$, where R is some ring. Here again, the field K may be arbitrary.

Corollary

The class of functors of finite length of $\mathcal{F}^{\mathrm{fd}}(\mathrm{P}(R); K)$ is stable under tensor product.

Global theorem: rôle of unipotents

Let F be a finitely-generated functor of $\mathcal{F}^{\mathrm{fd}}(\mathcal{A};K)$.
If x and y are objects of $\mathcal{A}$, we denote by $u[f]$ the automorphism of $x \oplus y$ whose components $x \to x$ and $y \to y$ are identities, $f : x \to y$ and $0 : y \to x$ (the letter u abbreviates the word *unipotent*).
A byproduct of the proof of our first theorem is the following:

Proposition

The functor F is Hom-polynomial (resp. antipolynomial) if and only if $F(u[f])$ is a unipotent (resp. absolutely semi-simple) automorphism for each morphism f of $\mathcal{A}$.

The multiplicative Jordan decomposition of the $F(u[f])$ will allow to get our decomposition à la Steinberg.

Global theorem: the K-cotrivial ideal

If x and y are objects of $\mathcal{A}$, we denote by $\mathcal{I}(x,y)$ the set of morphisms $f \in \mathcal{A}(x,y)$ such that, for each object t of $\mathcal{A}$, the automorphism $F(u[f] \oplus t)$ of $F(x \oplus y \oplus t)$ is unipotent.
The key step of the proof of the theorem is to prove the following result.

Lemma

$\mathcal{I}$ is a K-cotrivial ideal of $\mathcal{A}$

The intervention of '$\oplus t$' in the previous definition is essential to establish that $\mathcal{I}$ is an ideal of $\mathcal{A}$. But, once the lemma obtained, we will use only the unipotence of $F(u[f])$.

Continuation of the proof

Let us write the multiplicative Jordan decomposition as $F(u[f]) = U(f).D(f) = D(f).U(f)$, where $U(f)$ is unipotent and $D(f)$ is absolutely semi-simple.

We get a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(x,y) & \longrightarrow & \mathcal{A}(x,y) \times (\mathcal{A}/\mathcal{I})(x,y) \\ & \searrow^{f \mapsto F(u[f])} & \downarrow^{(f,\bar{g}) \mapsto D(f).U(g)} \\ & & \mathrm{Hom}_K(F(x \oplus y), F(x \oplus y)) \end{array}$$

(the top arrow being the canonical morphism of abelian groups and $\bar{g}$ denoting the class of $g \in \mathcal{A}(x,y)$ in $(\mathcal{A}/\mathcal{I})(x,y)$).

The factorisation of F

Then, we postcompose these functions with the linear map

$$\mathrm{Hom}_K(F(x\oplus y), F(x\oplus y)) \xrightarrow{\mathrm{Hom}_K(F(x\hookrightarrow x\oplus y), F(x\oplus y\twoheadrightarrow y))} \mathrm{Hom}_K(F(x), F(y))$$

which, precomposed with $f \mapsto F(u[f])$, gives the function

$$F_{x,y} : \mathcal{A}(x,y) \xrightarrow{f\mapsto F(f)} \mathrm{Hom}_K(F(x), F(y))\ ;$$

what gives the factorisation of F.

From unipotence to polynomiality

To prove the Hom-polynomial property with respect to the first variable of this factorisation, we use the following elementary lemma:

Lemma

Let M be an abelian group, V a K-vector space of finite dimension d and $\rho : M \to E := \mathrm{End}_K(V)$ a function such that $\rho(u+v) = \rho(u).\rho(v)$ for each $(u,v) \in M^2$. We suppose that ρ takes values in unipotent automorphisms of V. Then ρ defines a polynomial function of degree $\leq d-1$ from M to the underlying additive group of E.

Thank you for your attention!